Dirac structures for generalized Lie bialgebroids

2004 J. Phys. A: Math. Gen. 37 2671

(http://iopscience.iop.org/0305-4470/37/7/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2671–2692 PII: S0305-4470(04)66289-5

Dirac structures for generalized Lie bialgebroids

J M Nunes da Costa and J Clemente-Gallardo

Departamento de Matemática, Universidade de Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

E-mail: jmcosta@mat.uc.pt and jesus@mat.uc.pt

Received 15 July 2003, in final form 16 December 2003
Published 4 February 2004
Online at stacks.iop.org/JPhysA/37/2671 (DOI: 10.1088/0305-4470/37/7/011)

Abstract

We study Dirac structures for generalized Courant algebroids, which are doubles of generalized Lie bialgebroids. The cases investigated include graphs of bivector fields and characteristic pairs of some sub-bundles.

PACS numbers: 02.20.Sv, 02.40.Ma, 02.40.Vh
Mathematics Subject Classification: 17B62, 53D10, 53D17

1. Introduction

Dirac structures on manifolds were introduced by Courant and Weinstein [2] and studied in detail in [1]. Dirac structures include closed 2-forms, Poisson structures and foliations. Dorfman [3] developed an algebraic treatment of these structures and used them for the study of completely integrable systems, in the context of the calculus of variations.

The notion of Dirac structure on a manifold M, investigated by Courant in [1], is defined using a sub-bundle L of $TM \oplus T^*M$, which is maximally isotropic under the natural symmetric pairing on $TM \oplus T^*M$, and a bracket on the space of sections of $TM \oplus T^*M$, called the Courant bracket. The existence of a Dirac structure corresponds to the closeness of that bracket on the space $\Gamma(L)$ of sections of L.

In order to understand the meaning of this bracket, which is not a Lie bracket on $\Gamma(TM \oplus T^*M)$, Liu *et al* [17] introduced the notion of a Courant algebroid on a vector bundle whose definition includes a skew-symmetric bracket on the space of sections of that bundle. The first example of a Courant algebroid is the Whitney sum bundle $A \oplus A^*$, where the pair (A, A^*) is a Lie bialgebroid [19]. The Courant algebroid $A \oplus A^*$ is called the double of the Lie bialgebroid (A, A^*).

For the case of the Lie bialgebroid (TM, T^*M), where TM is the Lie algebroid whose space of sections is endowed with the usual Lie bracket of vector fields and T^*M is the null Lie algebroid, the bracket on the Courant algebroid $TM \oplus T^*M$ is that introduced in [1], i.e. the Courant bracket.

It is well known that there exists a close relation between Lie bialgebroids and Poisson structures on manifolds. However, if we pass from the Poisson to the Jacobi setting, we have to replace Lie bialgebroids by generalized Lie bialgebroids [6] or Jacobi bialgebroids [4]. In fact, in contrast to the Poisson case, the pair (TM, T^*M) is not, in general, a Lie bialgebroid if M is a Jacobi manifold. Canonically associated with a Jacobi manifold M, there is a generalized Lie bialgebroid structure on $(TM \times \mathbb{R}, T^*M \times \mathbb{R})$. In [26], Wade considered the Whitney sum bundle $\mathcal{E}^1(M) = (TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R})$ and introduced the notion of $\mathcal{E}^1(M)$-Dirac structure, extending the Courant bracket to the space of sections of $\mathcal{E}^1(M)$. But this extended bracket in not a Courant bracket.

One of the motivations for this paper was a tentative to understand the exact meaning of the extended bracket introduced by Wade, and to see how it is related to the Courant algebroid structure. For that, we introduce the notion of a generalized Courant algebroid, which is shown to include the double of a generalized Lie bialgebroid as a particular case. We then conclude that the bracket introduced by Wade is the bracket of a generalized Courant algebroid, defined on $\mathcal{E}^1(M)$, for the case where $T^*M \times \mathbb{R}$ is endowed with the null Lie algebroid structure.

In a very recent paper, Grabowski and Marmo [5] developed a theory of graded Jacobi brackets which unifies various concepts of graded Lie structures in geometry and physics. Among them, the notion of Courant–Jacobi algebroid is presented. This notion turns out to be equivalent to our definition of the generalized Courant algebroid, although our presentation is different from that of [5], since it is mainly based on the concept of generalized Lie bialgebroid introduced in [6].

Another purpose of this paper is the study of some Dirac structures for generalized Lie bialgebroids, such as graphs of bivector fields and characteristic pairs of some sub-bundles. The case of triangular generalized Lie bialgebroids is also investigated. In [5] the notion of Dirac structure associated with a Courant–Jacobi algebroid is also mentioned, although it is not exploited.

The paper is organized as follows. In section 2, we recall some definitions and results concerning generalized Lie bialgebroids and Jacobi manifolds. In section 3, we introduce the notion of generalized Courant algebroid, presenting two definitions: the first is a direct generalization of the definition of Courant algebroid in [17], and the second is obtained from the first, after reducing the number of axioms. We also prove that the notions of generalized Courant algebroid and Courant–Jacobi algebroid structures on a vector bundle are equivalent. In section 4, we prove that the double of a generalized Lie bialgebroid is a generalized Courant algebroid and we recover, as an example, the $\mathcal{E}^1(M)$-Dirac structure introduced by Wade in [26]. This result is referred to in [5], but it is obtained using different techniques. Sections 5 and 6 are devoted to the presentation of some examples of Dirac structures for generalized Lie bialgebroids $((A, \phi), (A^*, W))$. In section 5, we study the graph of a bivector field on A, i.e. a section of $\bigwedge^2 A$, and we investigate the case of a Jacobi structure $(\Lambda, E) \in \Gamma(\bigwedge^2(TM \times \mathbb{R}))$ on M. In section 6, we consider a sub-bundle $D \subset A$ and we establish the conditions that ensure the existence of a Dirac structure on the vector bundle $D \oplus D^\perp$, where $D^\perp$ is the conormal bundle of D. Using the notion of characteristic pair [16] of a sub-bundle A, we obtain another example of a Dirac structure. Finally, in section 7 we consider the Dirac structure for a triangular generalized Lie bialgebroid.

2. Generalized Lie bialgebroids and Jacobi structures

A *Lie algebroid* $(A, [\cdot,\cdot], \rho)$ over a manifold M is a vector bundle A over M together with a Lie bracket $[\cdot,\cdot]$ on the space $\Gamma(A)$ of the global cross sections of A and a bundle map

$\rho : A \to TM$, called the *anchor*, such that if we also denote by $\rho : \Gamma(A) \to \mathfrak{X}(M)$ the homomorphism induced by ρ, then

$$[X, fY] = f[X, Y] + (\rho(X)f)Y$$

for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M, \mathbb{R})$.

We note that if $(A, [\cdot,\cdot], \rho)$ is a Lie algebroid over M, then $\rho : (\Gamma(A), [\cdot,\cdot]) \to (\mathfrak{X}(M), [\cdot,\cdot])$ is a Lie algebra homomorphism.

Example 2.1. If M is a differentiable manifold, then the triple $(TM, [\cdot,\cdot], Id_{TM})$ is a Lie algebroid over M, where $[\cdot,\cdot]$ is the usual Lie bracket of vector fields. A Lie algebroid over a point is a Lie algebra.

Notation: Throughout this paper, we will use δ to denote the usual differential of de Rham for the manifold M.

Example 2.2. Let (M, Λ) be a Poisson manifold and $\Lambda^\# : T^*M \to TM$ the vector bundle morphism associated with the Poisson tensor Λ given, for any sections α, β of T^*M, by

$$\langle \beta, \Lambda^\#(\alpha)\rangle = \Lambda(\alpha, \beta).$$

Then the triple $(T^*M, [\cdot,\cdot]_\Lambda, \Lambda^\#)$ is a Lie algebroid over M, where $[\cdot,\cdot]_\Lambda$ is the Lie bracket of 1-forms given by

$$[\alpha, \beta]_\Lambda = \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)). \tag{1}$$

We recall that a *Jacobi structure* on a manifold M is a pair (Λ, E), where Λ is a bivector and E is a vector field such that $[\Lambda, \Lambda] = -2E \wedge \Lambda$ and $[E, \Lambda] = 0$, [15].

Example 2.3. Let (M, Λ, E) be a Jacobi manifold. We denote by $(\Lambda, E)^\# : T^*M \times \mathbb{R} \to TM \times \mathbb{R}$ the vector bundle morphism given by

$$(\Lambda, E)^\#(\alpha, f) = (\Lambda^\#(\alpha) + fE, -\langle \alpha, E\rangle) \tag{2}$$

for any section α of T^*M and $f \in C^\infty(M, \mathbb{R})$. In opposition to the case of a Poisson manifold, in general one cannot define a Lie algebroid structure on the cotangent bundle of a Jacobi manifold. However, if (M, Λ, E) is a Jacobi manifold, then $(T^*M \times \mathbb{R}, [\cdot,\cdot]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\#)$ is a Lie algebroid over M [10], where $\pi : TM \times \mathbb{R} \to TM$ is the projection over the first factor and $[\cdot,\cdot]_{(\Lambda,E)}$ is the bracket given by

$$[(\alpha, f), (\beta, g)]_{(\Lambda,E)} := (\gamma, r) \tag{3}$$

with

$$\gamma := \mathcal{L}_{\Lambda^\#(\alpha)}\beta - \mathcal{L}_{\Lambda^\#(\beta)}\alpha - \delta(\Lambda(\alpha, \beta)) + f\mathcal{L}_E\beta - g\mathcal{L}_E\alpha - i_E(\alpha \wedge \beta),$$
$$r := -\Lambda(\alpha, \beta) + \Lambda(\alpha, \delta g) - \Lambda(\beta, \delta f) + \langle f\delta g - g\delta f, E\rangle.$$

It is well known that, given a Lie algebroid $(A, [\cdot,\cdot], \rho)$, there exists an associated differential d on the graded space of sections of $\bigwedge A^* = \oplus_{k\in\mathbb{Z}} \bigwedge^k A^*$, where A^* is the dual vector bundle of A. More precisely, d is a derivation of degree 1 and of square 0 of the associative graded commutative algebra $(\Gamma(\bigwedge A^*), \wedge)$. Also the Lie bracket on $\Gamma(A)$ can be extended to the algebra of sections of $\bigwedge A$, $\Gamma(\bigwedge A) = \oplus_{k\in\mathbb{Z}}\Gamma(\bigwedge^k A)$. The result is a graded Lie bracket $[\cdot,\cdot]$ which is called the *Schouten bracket* of the Lie algebroid[1]. For more details, see [11, 18].

[1] Some differences in signs with [6, 15] come from different conventions for the Schouten bracket.

Let $(A, [\cdot,\cdot], \rho)$ be a Lie algebroid over M and $\phi \in \Gamma(A^*)$ a 1-cocycle for the Lie algebroid cohomology complex with trivial coefficients (see [6, 18]), i.e. for all $X, Y \in \Gamma(A)$,

$$\langle \phi, [X, Y]\rangle = \rho(X)(\langle \phi, Y\rangle) - \rho(Y)(\langle \phi, X\rangle). \tag{4}$$

Using the 1-cocycle ϕ, we can define a new representation ρ^ϕ of the Lie algebra $(\Gamma(A), [\cdot,\cdot])$ on $C^\infty(M, \mathbb{R})$, by setting

$$\rho^\phi : \Gamma(A) \times C^\infty(M, \mathbb{R}) \to C^\infty(M, \mathbb{R}) \qquad (X, f) \mapsto \rho^\phi(X, f) = \rho(X)f + \langle \phi, X\rangle f. \tag{5}$$

Therefore, we obtain a new cohomology complex, whose differential cohomology operator is given by

$$\mathrm{d}^\phi : \Gamma\big(\bigwedge^k A^*\big) \to \Gamma\big(\bigwedge^{k+1} A^*\big) \qquad \beta \mapsto \mathrm{d}^\phi(\beta) = \mathrm{d}\beta + \phi \wedge \beta. \tag{6}$$

Also, for any $X \in \Gamma(A)$, the Lie derivative operator with respect to X is given by

$$\mathcal{L}^\phi_X : \Gamma\big(\bigwedge^k A^*\big) \to \Gamma\big(\bigwedge^k A^*\big) \qquad \beta \mapsto \mathcal{L}^\phi_X(\beta) = \mathcal{L}_X\beta + \langle \phi, X\rangle\beta. \tag{7}$$

It is also possible to consider a ϕ-Schouten bracket on the graded algebra $\Gamma(\bigwedge A)$, denoted by $[\cdot,\cdot]^\phi$, which is defined as follows:

$$\begin{aligned}&[\cdot,\cdot]^\phi : \Gamma\big(\bigwedge^p A\big) \times \Gamma\big(\bigwedge^q A\big) \to \Gamma\big(\bigwedge^{p+q-1} A\big)\\ &(P, Q) \mapsto [P, Q]^\phi = [P, Q] + (p-1)P \wedge (i_\phi Q) + (-1)^p(q-1)(i_\phi P) \wedge Q\end{aligned} \tag{8}$$

where $i_\phi Q$ can be interpreted as the usual contraction of a multivector field by a 1-form. We observe that when $p = q = 1, [P, Q]^\phi = [P, Q]$. That is, the brackets $[\cdot,\cdot]^\phi$ and $[\cdot,\cdot]$ coincide on $\Gamma(A)$.

We can develop a differential calculus using ρ^ϕ, d^ϕ, $\mathcal{L}^\phi$ and $[\cdot,\cdot]^\phi$. The formulae obtained are similar, but adapted, to the case of a Lie algebroid (see [4, 6]).

Suppose that the vector bundle $(A, [\cdot,\cdot], \rho)$ and its dual vector bundle $(A^*, [\cdot,\cdot]_*, \rho_*)$ are both Lie algebroids over a manifold M. Let d (resp. d_*) denote the differential of A (resp. A^*). Let $\phi \in \Gamma(A^*)$ (resp. $W \in \Gamma(A)$) be a 1-cocycle in the Lie algebroid cohomology complex of $(A, [\cdot,\cdot], \rho)$ (resp. $(A^*, [\cdot,\cdot]_*, \rho_*)$).

Definition 2.4 ([6])**.** *The pair* $((A, \phi), (A^*, W))$ *is a generalized Lie bialgebroid if for all* $X, Y \in \Gamma(A)$ *and* $P \in \Gamma(\bigwedge^p A)$*, the following conditions hold:*

$$\mathrm{d}^W_*[X, Y] = [\mathrm{d}^W_* X, Y]^\phi + [X, \mathrm{d}^W_* Y]^\phi; \tag{9}$$

$$\mathcal{L}^W_{*\phi} P + \mathcal{L}^\phi_W P = 0. \tag{10}$$

Under the name of Jacobi bialgebroid, this notion was presented in [4], with the following definition:

Definition 2.5 ([4])**.** *The pair* $((A, \phi), (A^*, W))$ *is a Jacobi bialgebroid if for all* $P \in \Gamma(\bigwedge^p A)$ *and* $Q \in \Gamma(\bigwedge A)$*,*

$$\mathrm{d}^W_*[P, Q]^\phi = [\mathrm{d}^W_* P, Q]^\phi + (-1)^{p+1}[P, \mathrm{d}^W_* Q]^\phi. \tag{11}$$

The equivalence of definitions 2.4 and 2.5 was proved in [4].

When $\phi = 0$ and $W = 0$, we recover the notion of *Lie bialgebroid*: definition 2.4 generalizes the original definition introduced in [19] by Mackenzie and Xu, while definition 2.5 generalizes the equivalent one given by Kosmann-Schwarzbach [11].

The important property of duality of a Lie bialgebroid is also verified in the case of a generalized Lie bialgebroid: if $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid, so is

$((A^*, W), (A, \phi))$ (see [4, 6]). As a consequence, both definitions 2.4 and 2.5 can be given using the *dual versions* of (9)–(10) and (11), respectively.

Example 2.6. Let (M, Λ, E) be a Jacobi manifold. Consider the associated Lie algebroid $(T^*M \times \mathbb{R}, [\cdot,\cdot]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^\#)$ over M. Its differential d_* is given for all $(P, Q) \in \Gamma(\bigwedge^k(TM)) \oplus \Gamma(\bigwedge^{k+1}(TM))$, by ([14])

$$d_*(P, Q) = ([\Lambda, P] + kE \wedge P + \Lambda \wedge Q, -[\Lambda, Q] + (1-k)E \wedge Q + [E, P]). \tag{12}$$

On the other hand, if M is a differentiable manifold, then the triple $(TM \times \mathbb{R}, [\cdot,\cdot], \pi)$ is a Lie algebroid over M, where π is the projection over the first factor and $[\cdot,\cdot]$ is given by

$$[(X, f), (Y, g)] = ([X, Y], X(g) - Y(f)) \qquad (X, f), (Y, g) \in \mathfrak{X}(M) \times C^\infty(M, \mathbb{R}). \tag{13}$$

The associated differential is $d = (\delta, -\delta)$, δ being the de Rham differential.

In [6] it was proved that $\phi = (0, 1)$ (resp. $W = (-E, 0)$) is a 1-cocycle of $TM \times \mathbb{R}$ (resp. $T^*M \times \mathbb{R}$) and the pair $((TM \times \mathbb{R}, \phi), (T^*M \times \mathbb{R}, W))$ is a Jacobi bialgebroid.

Another interesting example of a generalized Lie bialgebroid is that provided by *strict Jacobi–Nijenhuis manifolds* (see [8, 22]).

When the base manifold of a generalized Lie bialgebroid reduces to a point, we obtain a *generalized Lie bialgebra* [6]. In other words, a generalized Lie bialgebra is a pair $((\mathcal{G}, \phi), (\mathcal{G}^*, W))$, where $(\mathcal{G}, [\cdot,\cdot])$ is a real Lie algebra of finite dimension such that

- the dual space $\mathcal{G}^*$ is also a Lie algebra with Lie bracket $[\cdot,\cdot]_*$,
- $\phi \in \mathcal{G}^*$ and $W \in \mathcal{G}$ are 1-cocycles on $\mathcal{G}$ and $\mathcal{G}^*$, respectively and
- $d_*^W[X, Y] = [d_*^W X, Y]^\phi + [X, d_*^W Y]^\phi$, $\langle \phi, W \rangle = 0$, $i_\phi(d_* X) + [W, X] = 0$, for all $X, Y \in \mathcal{G}$.

For more details on generalized Lie bialgebras, see [6, 9].

3. Generalized Courant algebroids and Courant–Jacobi algebroids

In this section, we introduce the notion of generalized Courant algebroid. We present first a definition that generalizes directly the definition of Courant algebroid introduced by Liu *et al* in [17]. Then, using [25], we reduce the number of axioms of this definition, to obtain a new version of it.

For the sake of completeness, we would also like to mention that a generalized Courant algebroid can be viewed as a particular case of a pure algebraic structure described in [24].

Very recently, the notion of Courant–Jacobi algebroid was introduced by Grabowski and Marmo in [5], based on the definition of Courant algebroid proposed by Roytenberg [23] and using some techniques of [25]. We show that the definitions of generalized Courant algebroid and Courant–Jacobi algebroid are equivalent.

Definition 3.1. *A generalized Courant algebroid is a pair (A, θ), where A is a vector bundle $A \to M$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(A)$ and a bundle map $\rho : A \to TM$, and $\theta \in \Gamma(A^*)$ is such that, for any $X, Y \in \Gamma(A)$, $\langle \theta, [X, Y] \rangle = \rho(X)\langle \theta, Y \rangle - \rho(Y)\langle \theta, X \rangle$, satisfying the following properties:*

1. *for any $X_1, X_2, X_3 \in \Gamma(A)$,*

$$[[X_1, X_2], X_3] + \text{c.p.} = \mathcal{D}^\theta T(X_1, X_2, X_3) \tag{14}$$

2. *for any* $X_1, X_2 \in \Gamma(A)$,

$$\rho([X_1, X_2]) = [\rho(X_1), \rho(X_2)] \tag{15}$$

3. *for any* $X_1, X_2 \in \Gamma(A)$ *and* $f \in C^\infty(M, \mathbb{R})$,

$$[X_1, fX_2] = f[X_1, X_2] + (\rho(X_1)f)X_2 - (X_1, X_2)\mathcal{D}f \tag{16}$$

4. *for any* $f, g \in C^\infty(M, \mathbb{R})$,

$$(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0 \tag{17}$$

5. *for any* $Y, X_1, X_2 \in \Gamma(A)$,

$$\begin{aligned}\rho(Y)(X_1, X_2) + \langle\theta, Y\rangle(X_1, X_2) = ([Y, X_1] + \mathcal{D}^\theta(Y, X_1), X_2)\\ + (X_1, [Y, X_2] + \mathcal{D}^\theta(Y, X_2))\end{aligned} \tag{18}$$

where $T(X_1, X_2, X_3)$ *is the function defined by*

$$T(X_1, X_2, X_3) = \tfrac{1}{3}([X_1, X_2], X_3) + \text{c.p.} \tag{19}$$

and $\mathcal{D}, \mathcal{D}^\theta : C^\infty(M, \mathbb{R}) \to \Gamma(A)$ *are given, for any* $X \in \Gamma(A)$, *respectively by*

$$(\mathcal{D}^\theta f, X) = \tfrac{1}{2}(\rho(X)f + \langle\theta, X\rangle f) \qquad \text{and} \qquad (\mathcal{D}f, X) = \tfrac{1}{2}\rho(X)f. \tag{20}$$

Remark 3.2. When $\theta = 0$, we recover the definition of *Courant algebroid*, introduced in [17].

If, for any $X \in \Gamma(A)$, we denote by $\rho^\theta(X)$ the first-order differential operator given by

$$\rho^\theta(X) = \rho(X) + \langle\theta, X\rangle \tag{21}$$

we can replace (18) by

$$\rho^\theta(Y)(X_1, X_2) = ([Y, X_1] + \mathcal{D}^\theta(Y, X_1), X_2) + (X_1, [Y, X_2] + \mathcal{D}^\theta(Y, X_2)). \tag{22}$$

Moreover, (15) is equivalent to

$$\rho^\theta([X_1, X_2]) = [\rho^\theta(X_1), \rho^\theta(X_2)] \tag{23}$$

where the bracket on the right-hand side is the Lie bracket (13) on $\mathfrak{X}(M) \times C^\infty(M, \mathbb{R})$.

Following the ideas of Uchino [25] let us see that it is possible to reduce the number of axioms in definition 3.1. Let us assume that conditions 1, 2 and 5 in definition 3.1 hold, where $\mathcal{D}^\theta$ is a map from $C^\infty(M, \mathbb{R})$ to $\Gamma(A)$, $\mathcal{D}^\theta : C^\infty(M, \mathbb{R}) \to \Gamma(A)$, which is a first-order differential operator, i.e.,

$$\mathcal{D}^\theta(fg) = f\mathcal{D}^\theta g + g\mathcal{D}^\theta f - fg\mathcal{D}^\theta 1 \qquad \forall f, g \in C^\infty(M, \mathbb{R}).$$

By $\mathcal{D} : C^\infty(M, \mathbb{R}) \to \Gamma(A)$ we denote the derivation associated with the first-order differential operator $\mathcal{D}^\theta$, that is $\mathcal{D}f = \mathcal{D}^\theta f - f\mathcal{D}^\theta 1$, $f \in C^\infty(M, \mathbb{R})$. Under these conditions, let us prove the following lemma.

Lemma 3.3. *For any* $X_1, X_2, Y \in \Gamma(A)$ *and* $f \in C^\infty(M, \mathbb{R})$,

$$[X_1, fX_2] = f[X_1, X_2] + (\rho(X_1)f)X_2 - (X_1, X_2)\mathcal{D}f. \tag{24}$$

Proof. From condition 5 in definition 3.1, and using (22), we have

$$\rho^\theta(X_1)(fX_2, Y) = ([X_1, fX_2] + \mathcal{D}^\theta(X_1, fX_2), Y) + (fX_2, [X_1, Y] + \mathcal{D}^\theta(X_1, Y)). \tag{25}$$

But, since $\rho^\theta(X_1)$ is a first-order differential operator,

$$\begin{aligned}\rho^\theta(X_1)(fX_2, Y) &= f\rho^\theta(X_1)(X_2, Y) + (\rho^\theta(X_1)f)(X_2, Y) - f(X_2, Y)\langle\theta, X_1\rangle\\ &= (f[X_1, X_2] + f\mathcal{D}^\theta(X_1, X_2), Y) + (fX_2, [X_1, Y] + \mathcal{D}^\theta(X_1, Y))\\ &\quad + (X_2, Y)(\rho(X_1)f)\end{aligned} \tag{26}$$

and, taking into account that $(\cdot,\cdot)$ is nondegenerate, we obtain

$$[X_1, fX_2] + \mathcal{D}^\theta(X_1, fX_2) = f[X_1, X_2] + f\mathcal{D}^\theta(X_1, X_2) + (\rho(X_1)f)X_2. \tag{27}$$

The equality

$$\begin{aligned}\mathcal{D}^\theta(X_1, fX_2) &= f\mathcal{D}^\theta(X_1, X_2) + (X_1, X_2)\mathcal{D}^\theta f - f(X_1, X_2)\mathcal{D}^\theta 1\\ &= f\mathcal{D}^\theta(X_1, X_2) + (X_1, X_2)\mathcal{D}f\end{aligned}$$

and (27) lead directly to (24). □

Now we can give the following definition of generalized Courant algebroid.

Definition 3.4. *A generalized Courant algebroid is a pair (A, θ), where A is a vector bundle $A \to M$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle, a skew-symmetric bracket $[\cdot,\cdot]$ on $\Gamma(A)$ and a bundle map $\rho^\theta : A \to TM \times \mathbb{R}$, which is a first-order differential operator, and where $\theta \in \Gamma(A^*)$ is such that, for any $X, Y \in \Gamma(A)$, $\langle\theta, [X, Y]\rangle = \rho(X)\langle\theta, Y\rangle - \rho(Y)\langle\theta, X\rangle$, $\rho(X)$ being the derivation associated with $\rho^\theta(X)$ (i.e., $\rho^\theta(X) = \rho(X) + \langle\theta, X\rangle$), satisfying the following properties:*

(i) for any $X_1, X_2, X_3 \in \Gamma(A)$,

$$[[X_1, X_2], X_3] + \text{c.p.} = \mathcal{D}^\theta T(X_1, X_2, X_3) \tag{28}$$

(ii) for any $X_1, X_2 \in \Gamma(A)$,

$$\rho^\theta([X_1, X_2]) = [\rho^\theta(X_1), \rho^\theta(X_2)] \tag{29}$$

where the bracket on the right-hand side is the usual Lie bracket on $\Gamma(TM \times \mathbb{R})$;

(iii) for any $Y, X_1, X_2 \in \Gamma(A)$,

$$\rho^\theta(Y)(X_1, X_2) = ([Y, X_1] + \mathcal{D}^\theta(Y, X_1), X_2) + (X_1, [Y, X_2] + \mathcal{D}^\theta(Y, X_2)) \tag{30}$$

(iv) for any $f, g \in C^\infty(M, \mathbb{R})$,

$$(\mathcal{D}^\theta f, \mathcal{D}^\theta g) = 0 \tag{31}$$

where $T(X_1, X_2, X_3)$ is the function defined by

$$T(X_1, X_2, X_3) = \tfrac{1}{3}([X_1, X_2], X_3) + \text{c.p.} \tag{32}$$

and $\mathcal{D}^\theta : C^\infty(M, \mathbb{R}) \to \Gamma(A)$ is the first-order differential operator given, for all $Y \in \Gamma(A)$, by

$$(\mathcal{D}^\theta f, Y) = \tfrac{1}{2}\rho^\theta(Y)f. \tag{33}$$

In [5] the authors introduced the following definition of Courant–Jacobi algebroid.

Definition 3.5. *A Courant–Jacobi algebroid is a vector bundle A over M together with*

1. a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on the bundle;

2. a blilinear operation $\circ$ on $\Gamma(A)$ such that, for any $X_1, X_2, X_3 \in \Gamma(A)$,

$$X_1 \circ (X_2 \circ X_3) = (X_1 \circ X_2) \circ X_3 + X_2 \circ (X_1 \circ X_3) \tag{34}$$

3. a bundle map $\lambda : A \to TM \times \mathbb{R}$ which is a homomorphism into the Lie algebroid of first-order differential operators:

$$\lambda(X \circ Y) = [\lambda(X), \lambda(Y)] \qquad \forall X, Y \in \Gamma(A) \tag{35}$$

satisfying the following properties:

(a) $(Y \circ X, X) = (Y, X \circ X)$,
(b) $\lambda(X)(Y, Y) = 2(X \circ Y, Y)$,

for all $X, Y \in \Gamma(A)$.

It is noted in [5] that this definition can be formulated in terms of the first-order differential operator $\mathrm{D} : C^\infty(M, \mathbb{R}) \to \Gamma(A)$ given by

$$(\mathrm{D}(f), X) = \tfrac{1}{2}\lambda(X) f. \tag{36}$$

The next two propositions establish the equivalence of definitions 3.4 and 3.5.

Proposition 3.6. *If $A \to M$ is a Courant–Jacobi algebroid, it is a generalized Courant algebroid.*

Proof. Let $[\cdot, \cdot]$ denote the skew-symmetrization of the operation $\circ$ on $\Gamma(A)$, i.e.,

$$[X_1, X_2] = \tfrac{1}{2}(X_1 \circ X_2 - X_2 \circ X_1) \qquad X_1, X_2 \in \Gamma(A).$$

From (36) we obtain, for all $X, Y \in \Gamma(A)$,

$$\lambda(Y)(X, X) = 2(\mathrm{D}(X, X), Y)$$

and using (a) and (b) in definition 3.5, we write

$$\lambda(Y)(X, X) = 2(Y \circ X, X) = 2(Y, X \circ X).$$

Taking into account that $(\cdot, \cdot)$ is nondegenerate, we get

$$X \circ X = \mathrm{D}(X, X),$$

which implies that, for all $X_1, X_2 \in \Gamma(A)$,

$$\tfrac{1}{2}(X_1 \circ X_2 + X_2 \circ X_1) = \mathrm{D}(X_1, X_2)$$

and so

$$X_1 \circ X_2 = [X_1, X_2] + \mathrm{D}(X_1, X_2).$$

We equip $\Gamma(A)$ with this skew-symmetric bracket $[\cdot, \cdot]$.

Let $p : TM \times \mathbb{R} \to TM$ denote the canonical projection over the first factor. We take $\rho = p \circ \lambda : A \to TM$ and $\theta = \lambda^*((0, 1)) \in \Gamma(A^*)$, where λ^* is the transpose of λ and define, for any $X \in \Gamma(A)$,

$$\begin{aligned}\rho^\theta(X) &= \rho(X) + \langle \theta, X \rangle \\ &= p(\lambda(X)) + \langle (0, 1), \lambda(X) \rangle \\ &= \lambda(X).\end{aligned} \tag{37}$$

In other words, $i_\theta = \lambda - \rho$ and $\lambda = \rho^\theta$.

Let us observe that condition 3 in definition 3.5 and the fact that $(0, 1) \in \Gamma(T^*M \times \mathbb{R}) \equiv \Omega^1(M) \times C^\infty(M, \mathbb{R})$ is a 1-cocycle for the Lie algebroid $TM \times \mathbb{R}$ over M, ensure that, for any $X, Y \in \Gamma(A)$,

$$\langle \theta, [X, Y] \rangle = \rho(X)\langle \theta, Y \rangle - \rho(Y)\langle \theta, X \rangle.$$

At this stage, it is obvious that we must take $\mathcal{D}^\theta = \mathrm{D}$.

Let us now prove that (i), (ii), (iii) and (iv) in definition 3.4 hold. From (b) in definition 3.5 we deduce that, for all $X, Y_1, Y_2 \in \Gamma(A)$

$$\lambda(X)(Y_1, Y_2) = (X \circ Y_1, Y_2) + (X \circ Y_2, Y_1)$$

and by the very definition of $\circ$, we obtain (iii). Now, condition 3 in definition 3.5 gives

$$\rho^\theta(X \circ X) = 0 \qquad \forall X \in \Gamma(A),$$

which implies $\rho^\theta \circ \mathcal{D}^\theta = 0$. So,

$$\rho^\theta(X_1 \circ X_2) = \rho^\theta([X_1, X_2]) \qquad \forall X_1, X_2 \in \Gamma(A)$$

and (ii) holds. From $\rho^\theta \circ \mathcal{D}^\theta = 0$, we also obtain (iv).

Finally, to prove (i),

$$[[X_1, X_2], X_3] + \text{c.p.} = \mathcal{D}^\theta T(X_1, X_2, X_3) \qquad \forall X_1, X_2, X_3 \in \Gamma(A)$$

we do a computation as in [23]:

$$\begin{aligned} 0 &= (X_1 \circ X_2) \circ X_3 + X_2 \circ (X_1 \circ X_3) - X_1 \circ (X_2 \circ X_3) \\ &= [[X_1, X_2], X_3] + [[X_3, X_1], X_2] + [[X_2, X_3], X_1] \\ &\quad + [\mathcal{D}^\theta(X_1, X_2), X_3] + [X_2, \mathcal{D}^\theta(X_1, X_3)] - [X_1, \mathcal{D}^\theta(X_2, X_3)] \\ &\quad + \mathcal{D}^\theta(X_1 \circ X_2, X_3) + \mathcal{D}^\theta(X_1 \circ X_3, X_2) - \mathcal{D}^\theta(X_2 \circ X_3, X_1). \end{aligned} \tag{38}$$

The proof is complete if we show that the sum of the last six terms of (38), that we denote by $A(X_1, X_2, X_3)$, equals $-\mathcal{D}^\theta T(X_1, X_2, X_3)$. For that, we deduce from (b) that, for all $X \in \Gamma(A)$, $f \in C^\infty(M, \mathbb{R})$:

$$X \circ \mathcal{D}^\theta f = 2\mathcal{D}^\theta(\mathcal{D}^\theta f, X)$$

and also

$$\mathcal{D}^\theta f \circ X = 0,$$

which gives

$$[X, \mathcal{D}^\theta f] = \mathcal{D}^\theta(\mathcal{D}^\theta f, X). \tag{39}$$

Then,

$$\begin{aligned} A(X_1, X_2, X_3) &= \mathcal{D}^\theta([X_1, X_2], X_3) + \mathcal{D}^\theta(X_2, [X_1, X_3]) - \mathcal{D}^\theta(X_1, [X_2, X_3]) \\ &\quad + 2\mathcal{D}^\theta(X_2, \mathcal{D}^\theta(X_1, X_3)) - 2\mathcal{D}^\theta(X_1, \mathcal{D}^\theta(X_2, X_3)). \end{aligned} \tag{40}$$

A similar computation as in [23] shows that $A(X_1, X_2, X_3)$ is completely skew-symmetric (this proof needs condition (a) in definition 3.5). The fact that A equals its skew-symmetrization gives the equality $A(X_1, X_2, X_3) = -\mathcal{D}^\theta T(X_1, X_2, X_3)$. □

Proposition 3.7. *If $A \to M$ is a generalized Courant algebroid, then it is a Courant–Jacobi algebroid.*

Proof. Let A be a generalized Courant algebroid and let us define the following operation on $\Gamma(A)$:

$$X_1 \circ X_2 = [X_1, X_2] + \mathcal{D}^\theta(X_1, X_2) \qquad \forall X_1, X_2 \in \Gamma(A).$$

We take $\lambda = \rho^\theta$; then $\mathrm{D} = \mathcal{D}^\theta$. Since we have $\rho^\theta \circ \mathcal{D}^\theta = 0$ in a generalized Courant algebroid, it is obvious that (ii) in definition 3.4 implies 3 in definition 3.5.

In the proof of proposition 3.6, we showed that

$$\begin{aligned} X_1 \circ (X_2 \circ X_3) &= (X_1 \circ X_2) \circ X_3 + X_2 \circ (X_1 \circ X_3) \\ &\Rightarrow [[X_1, X_2], X_3] + \text{c.p.} = \mathcal{D}^\theta T(X_1, X_2, X_3). \end{aligned}$$

If we look at the technical details of this proof we see that the implication $\Leftarrow$ also holds.

It remains to show that (a) and (b) in definition 3.5 hold. From (iii) in definition 3.4,

$$\rho^\theta(Y)(X,X) = ([Y,X] + \mathcal{D}^\theta(Y,X), X) + (X, [Y,X] + \mathcal{D}^\theta(Y,X))$$
$$= 2(Y \circ X, X) \tag{41}$$

which is exactly (b).

Finally,

$$(Y \circ X, X) = ([Y,X] + \mathcal{D}^\theta(Y,X), X) \overset{b)}{=} \tfrac{1}{2}\rho^\theta(Y)(X,X)$$
$$\overset{(36)}{=} (\mathcal{D}^\theta(X,X), Y) = (X \circ X, Y) \tag{42}$$

and (a) holds. □

A sub-bundle $L \subset A$ of the generalized Courant algebroid (A,θ) is said to be *integrable* if $\Gamma(L)$ is closed under the bracket $[\cdot,\cdot]$.

Definition 3.8. *A Dirac structure for the generalized Courant algebroid (A,θ) is an integrable sub-bundle L of A which is maximally isotropic with respect to the symmetric bilinear form $(\cdot,\cdot)$.*

An immediate consequence of the previous definition is the following.

Proposition 3.9. *If L is a Dirac structure for the generalized Courant algebroid (A,θ) and $\theta \in \Gamma(L^*)$, then $(L, \rho|_L, [\cdot,\cdot]|_L)$ is a Lie algebroid and θ is a 1-cocycle for the Lie algebroid cohomology complex with trivial coefficients.*

4. Generalized Courant algebroids and generalized Lie bialgebroids

In this section, we will see that the double of a generalized Lie bialgeboid is a generalized Courant algebroid. This result is referred to in [5] but it is obtained using different techniques.

Suppose that the vector bundle $A \to M$ and its dual $A^* \to M$ are both equipped with Lie algebroid structures $([\cdot,\cdot], a)$ and $([\cdot,\cdot]_*, a_*)$, respectively. Let d (resp. d_*) denote the differential of A (resp. A^*) and let $\phi \in \Gamma(A^*)$ (resp. $W \in \Gamma(A)$) be a 1-cocycle in the Lie algebroid cohomology complex of $(A, [\cdot,\cdot], a)$ (resp. $(A^*, [\cdot,\cdot]_*, a_*)$). Moreover, let d^ϕ, $\mathcal{L}^\phi$ and $[\cdot,\cdot]^\phi$ (resp. d_*^W, $\mathcal{L}_*^W$ and $[\cdot,\cdot]_*^W$) be the differential, the Lie derivative and the bracket modified by the 1-cocycle ϕ of A (resp. W of A^*) given by (6), (7) and (8), respectively.

On the Whitney sum bundle $A \oplus A^*$ we can define two nondegenerate bilinear forms, one symmetric, denoted by $(\cdot,\cdot)_+$, and the other skew-symmetric, denoted by $(\cdot,\cdot)_-$, by setting, for any $X_1 + \alpha_1, X_2 + \alpha_2 \in A \oplus A^*$,

$$(X_1 + \alpha_1, X_2 + \alpha_2)_+ = \tfrac{1}{2}(\langle \alpha_1, X_2\rangle + \langle \alpha_2, X_1\rangle) \tag{43}$$

and

$$(X_1 + \alpha_1, X_2 + \alpha_2)_- = \tfrac{1}{2}(\langle \alpha_1, X_2\rangle - \langle \alpha_2, X_1\rangle) \tag{44}$$

respectively.

On the space $\Gamma(A \oplus A^*)$ of the global cross sections of $A \oplus A^*$, which is identified with $\Gamma(A) \oplus \Gamma(A^*)$, we consider the following bracket (see [5]):

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = \left([X_1, X_2]^\phi + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1 - \mathrm{d}_*^W (e_1, e_2)_-\right)$$
$$+ \left([\alpha_1, \alpha_2]_*^W + \mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1 + \mathrm{d}^\phi(e_1, e_2)_-\right) \tag{45}$$

where $e_1 = X_1 + \alpha_1$ and $e_2 = X_2 + \alpha_2$.

Using the anchor maps a and a_*, and the 1-cocycles ϕ and W, we define the vector bundle maps $\rho : A \oplus A^* \to TM$ and $\rho^{\phi+W} : A \oplus A^* \to TM \times \mathbb{R}$, which are given, for any section

$X + \alpha$ of $A \oplus A^*$, by

$$\rho(X+\alpha) = a(X) + a_*(\alpha) \qquad \rho^{\phi+W}(X+\alpha) = a(X) + a_*(\alpha) + \langle \phi, X\rangle + \langle \alpha, W\rangle \tag{46}$$

respectively.

Proposition 4.1. *If $((A,\phi),(A^*,W))$ is a generalized Lie bialgebroid over M, then the pair $(A\oplus A^*,\theta)$, with $\theta = \phi + W$ is a generalized Courant algebroid with the Lie bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(A\oplus A^*)$ given by (45), the symmetric bilinear form given by (43), the vector bundle map ρ^θ given by (46) and the operator $\mathcal{D}^\theta$ given by $\mathcal{D}^\theta = (\mathrm{d}^\phi + \mathrm{d}_*^W)_{|C^\infty(M,\mathbb{R})}$.*

Before proving this proposition, we need the results of the following lemma.

Lemma 4.2. *Let $((A,\phi),(A^*,W))$ be a generalized Lie bialgebroid over M. Then, for any $X\in\Gamma(A)$, $\alpha\in\Gamma(A^*)$ and $f,g\in C^\infty(M,\mathbb{R})$,*

(i)

$$\begin{aligned} &\langle\phi, W\rangle = 0 && a(W) + a_*(\phi) = 0\\ &\mathcal{L}_{*\phi}X + [W,X] = 0 && \mathcal{L}_W\alpha + [\phi,\alpha]_* = 0\end{aligned} \tag{47}$$

(ii)

$$[\mathrm{d}_*^W f, X]^\phi + \mathcal{L}^W_{*\mathrm{d}^\phi f}X = 0 \qquad [\mathrm{d}^\phi f,\alpha]_*^W + \mathcal{L}^\phi_{\mathrm{d}_*^W f}\alpha = 0 \tag{48}$$

(iii)

$$(a\circ \mathrm{d}_*^W + a_*\circ \mathrm{d}^\phi) f = 0 \tag{49}$$

(iv)

$$[a(X), a_*(\alpha)] = a_*\big(\mathcal{L}_X^\phi\alpha\big) - a\big(\mathcal{L}^W_{*\alpha}X\big) + a(\mathrm{d}_*^W(\langle\alpha, X\rangle)) \tag{50}$$

(v)

$$\langle \mathrm{d}^\phi f, \mathrm{d}_*^W g\rangle + \langle \mathrm{d}^\phi g, \mathrm{d}_*^W f\rangle = 0. \tag{51}$$

Proof. Conditions (i) and (ii) and (v) were proved in [6], while (iii) can be deduced from (ii). For (iv), we have

$$\begin{aligned}
&[a(X), a_*(\alpha)](f) - a_*\big(\mathcal{L}_X^\phi\alpha\big)f + a\big(\mathcal{L}^W_{*\alpha}X\big)f\\
&\quad= a(X)(\langle\alpha, \mathrm{d}_* f\rangle) - \big\langle \mathcal{L}_X^\phi\alpha, \mathrm{d}_* f\big\rangle - a_*(\alpha)\langle \mathrm{d}f, X\rangle + \big\langle \mathrm{d}f, \mathcal{L}^W_{*\alpha}X\big\rangle\\
&\quad= \langle\alpha, [X, \mathrm{d}_* f]\rangle - \langle\phi, X\rangle(a_*(\alpha)f) - \langle[\alpha, \mathrm{d}f]_*, X\rangle + \langle\alpha, X\rangle(a(X)f)\\
&\quad= \langle\alpha, [X, \mathrm{d}_*^W f]\rangle - f\langle\alpha, [X, W]\rangle + \langle\mathcal{L}_{*\mathrm{d}f}\alpha, X\rangle - \langle\phi, X\rangle(a_*(\alpha)f)\\
&\quad\overset{(48)}{=} \big\langle\alpha, \mathcal{L}^W_{*\mathrm{d}^\phi f}X\big\rangle - f\langle\alpha, [X, W]\rangle + \langle\mathcal{L}_{*\mathrm{d}f}\alpha, X\rangle - \langle\phi, X\rangle(a_*(\alpha)f)\\
&\quad= \mathcal{L}_{*\mathrm{d}f}(\langle\alpha, X\rangle) + \langle\alpha, X\rangle(a(X)f) + f\langle\alpha, \underbrace{\mathcal{L}_{*\phi}X - [X, W]}_{=0}\rangle\\
&\quad= \mathcal{L}^W_{*\mathrm{d}f}(\langle\alpha, X\rangle) = a(\mathrm{d}_*^W\langle\alpha, X\rangle)f.
\end{aligned}$$

□

Proof of proposition 4.1. First we note that, since $\phi\in\Gamma(A^*)$ and $W\in\Gamma(A)$ are 1-cocycles of A and A^* respectively, and using (i) of lemma 4.2, a straightforward computation shows that, with $\theta = \phi + W$ and $\rho = a + a_*$,

$$\langle\theta, [\![X_1+\alpha_1, X_2+\alpha_2]\!]\rangle = \rho(X_1+\alpha_1)\langle\theta, X_2+\alpha_2\rangle - \rho(X_2+\alpha_2)\langle\theta, X_1+\alpha_1\rangle \tag{52}$$

holds for all $X_1+\alpha_1, X_2+\alpha_2 \in \Gamma(A\oplus A^*)$. Moreover, the operator $\mathcal{D}^\theta = (\mathrm{d}^\phi + \mathrm{d}_*^W)_{|C^\infty(M,\mathbb{R})}$ is obviously a first-order differential operator.

Next, we show that all conditions of definition 3.4 hold.

(i) The proof of condition (i) of definition 3.4 involves a very long computation. We only give a short schedule, following the ideas of [17]. Let $e_i = X_i + \alpha_i$, $i = 1, 2, 3$, be any sections of $A \oplus A^*$. Then,

$$
\begin{aligned}
(\llbracket e_1, e_2 \rrbracket, e_3)_+ = & \big[\tfrac{1}{2}(\langle \alpha_3, [X_1, X_2]\rangle + \langle [\alpha_1, \alpha_2]_*, X_3\rangle + a(X_3)(e_1, e_2)_- \\
& - a_*(\alpha_3)(e_1, e_2)_- + (\langle \phi, X_3\rangle - \langle \alpha_3, W\rangle)(e_1, e_2)_-) + \text{c.p.}\big] \\
& + \tfrac{1}{2}(a(e_1) + a_*(e_1))(e_2, e_3)_+ - \tfrac{1}{2}(a(e_2) + a_*(e_2))(e_3, e_1)_+ \\
& + \tfrac{1}{2}(\langle \phi, X_1\rangle + \langle \alpha_1, W\rangle)(e_2, e_3)_+ - \tfrac{1}{2}(\langle \phi, X_2\rangle + \langle \alpha_2, W\rangle)(e_3, e_1)_+ \\
= & T(e_1, e_2, e_3) + \tfrac{1}{2}\rho(e_1)(e_2, e_3)_+ - \tfrac{1}{2}\rho(e_2)(e_3, e_1)_+ \\
& + \tfrac{1}{2}\langle \theta, e_1\rangle(e_2, e_3)_+ - \tfrac{1}{2}\langle \theta, e_2\rangle(e_3, e_1)_+. \qquad (53)
\end{aligned}
$$

Furthermore, we can prove the following equality:

$$
\begin{aligned}
(\llbracket e_1, e_2 \rrbracket, e_3)_- + \text{c.p.} = & T(e_1, e_2, e_3) + [(a(X_3)(e_1, e_2)_- + 2a_*(\alpha_3)(e_1, e_2)_- \\
& - \langle [\alpha_1, \alpha_2]_*, X_3\rangle + (\langle \phi, X_3\rangle + 2\langle \alpha_3, W\rangle)(e_1, e_2)_-) + \text{c.p.}]. \qquad (54)
\end{aligned}
$$

Let us set

$$\llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \text{c.p.} = Y + \beta$$

where Y (resp. β) stands for the part of $\llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket$ + c.p. that belongs to $\Gamma(A)$ (resp. $\Gamma(A^*)$). Using the formula ([4])

$$\mathcal{L}^\phi_X \circ \mathcal{L}^\phi_Y = \mathcal{L}^\phi_{[X,Y]} \qquad \forall X, Y \in \Gamma(A) \qquad (55)$$

we deduce

$$
\begin{aligned}
\beta = & \big\{\big[\mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1, \alpha_3\big]_* + [\mathrm{d}^\phi(e_1, e_2)_-, \alpha_3]_* + \mathcal{L}^\phi_{\mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1}\alpha_3 \\
& - \mathcal{L}^\phi_{\mathrm{d}^W_*(e_1,e_2)_-}\alpha_3 - \mathcal{L}^\phi_{X_3}[\alpha_1, \alpha_2]_* + \mathrm{d}^\phi(\llbracket e_1, e_2 \rrbracket, e_3)_- \\
& - \mathrm{d}^\phi(a(X_3)(e_1, e_2)_-) - \mathrm{d}^\phi(\langle \phi, X_3\rangle(e_1, e_2)_-)\big\} + \text{c.p.} \\
= & \big\{\mathrm{d}^\phi[(\llbracket e_1, e_2 \rrbracket, e_3)_- - (a(X_3)(e_1, e_2)_-) - 2(a_*(\alpha_3)(e_1, e_2)_-) \\
& + \langle [\alpha_1, \alpha_2]_*, X_3\rangle - (\langle \phi, X_3\rangle + 2\langle \alpha_3, W\rangle)(e_1, e_2)_-)] - i_{X_3}\big(\mathrm{d}^\phi[\alpha_1, \alpha_2]_* \\
& - \mathcal{L}^W_{*\alpha_1}\mathrm{d}^\phi\alpha_2 + \mathcal{L}^W_{*\alpha_2}\mathrm{d}^\phi\alpha_1\big) - [\mathrm{d}^\phi(e_1, e_2)_-, \alpha_3]_* - \mathcal{L}^\phi_{\mathrm{d}^W_*(e_1,e_2)_-}\alpha_3\big\} + \text{c.p.} \qquad (56)
\end{aligned}
$$

By lemma 4.2 (ii),

$$[\mathrm{d}^\phi(e_1, e_2)_-, \alpha_3]_* + \mathcal{L}^\phi_{\mathrm{d}^W_*(e_1,e_2)_-}\alpha_3 = 0.$$

Moreover,

$$\mathrm{d}^\phi[\alpha_1, \alpha_2]_* - \mathcal{L}^W_{*\alpha_1}\mathrm{d}^\phi\alpha_2 + \mathcal{L}^W_{*\alpha_2}\mathrm{d}^\phi\alpha_1 = \mathrm{d}^\phi[\alpha_1, \alpha_2]^W_* - [\alpha_1, \mathrm{d}^\phi\alpha_2]^W_* + [\alpha_2, \mathrm{d}^\phi\alpha_1]^W_* = 0$$

by the definition of generalized Lie bialgebroid.

So, we obtain

$$
\begin{aligned}
\beta = & \mathrm{d}^\phi[(\llbracket e_1, e_2 \rrbracket, e_3)_- - (a(X_3)(e_1, e_2)_-) - 2(a_*(\alpha_3)(e_1, e_2)_-) + \langle [\alpha_1, \alpha_2]_*, X_3\rangle \\
& - (\langle \phi, X_3\rangle + 2\langle \alpha_3, W\rangle)(e_1, e_2)_-)] + \text{c.p.} \\
\overset{(54)}{=} & \mathrm{d}^\phi(T(e_1, e_2, e_3)). \qquad (57)
\end{aligned}
$$

Similarly, one has

$$Y = \mathrm{d}^W_*(T(e_1, e_2, e_3)). \qquad (58)$$

From (57) and (58), we conclude that

$$\llbracket \llbracket e_1, e_2 \rrbracket, e_3 \rrbracket + \text{c.p.} = \mathcal{D}^\theta T(e_1, e_2, e_3).$$

(ii) For any $e_1 = X_1 + \alpha_1, e_2 = X_2 + \alpha_2 \in \Gamma(A \oplus A^*)$, we compute

$$\begin{aligned}
\rho^\theta([\![X_1+\alpha_1, X_2+\alpha_2]\!]) &= \rho([\![X_1+\alpha_1, X_2+\alpha_2]\!]) + \langle\theta, [\![X_1+\alpha_1, X_2+\alpha_2]\!]\rangle\\
&= a([X_1, X_2] + \mathcal{L}^W_{*\alpha_1}X_2 - \mathcal{L}^W_{*\alpha_2}X_1 - \mathrm{d}^W_*(e_1, e_2)_-)\\
&\quad + a_*\big([\alpha_1, \alpha_2]_* + \mathcal{L}^\phi_{X_1}\alpha_2 - \mathcal{L}^\phi_{X_2}\alpha_1 + \mathrm{d}^\phi(e_1, e_2)_-\big) + \langle\theta, [\![X_1+\alpha_1, X_2+\alpha_2]\!]\rangle\\
&= [a(X_1), a(X_2)] + [a_*(\alpha_1), a_*(\alpha_2)] + \big\{a\big(\mathcal{L}^W_{*\alpha_1}X_2\big) - a_*\big(\mathcal{L}^\phi_{X_2}\alpha_1\big)\\
&\quad - \tfrac{1}{2}a(\mathrm{d}^W_*\langle\alpha_1, X_2\rangle) + \tfrac{1}{2}a_*(\mathrm{d}^\phi\langle\alpha_1, X_2\rangle)\big\} - \big\{a\big(\mathcal{L}^W_{*\alpha_2}X_1\big) - a_*\big(\mathcal{L}^\phi_{X_1}\alpha_2\big)\\
&\quad - \tfrac{1}{2}a(\mathrm{d}^W_*\langle\alpha_2, X_1\rangle) + \tfrac{1}{2}a_*(\mathrm{d}^\phi\langle\alpha_2, X_1\rangle)\big\} + \langle\theta, [\![X_1+\alpha_1, X_2+\alpha_2]\!]\rangle\\
&\overset{(49)}{=} [a(X_1), a(X_2)] + [a_*(\alpha_1), a_*(\alpha_2)]\\
&\quad + \big\{a\big(\mathcal{L}^W_{*\alpha_1}X_2\big) - a_*\big(\mathcal{L}^\phi_{X_2}\alpha_1\big) - a(\mathrm{d}^W_*\langle\alpha_1, X_2\rangle)\big\}\\
&\quad - \big\{a\big(\mathcal{L}^W_{*\alpha_2}X_1\big) - a_*\big(\mathcal{L}^\phi_{X_1}\alpha_2\big) - a(\mathrm{d}^W_*\langle\alpha_2, X_1\rangle)\big\} + \langle\theta, [\![X_1+\alpha_1, X_2+\alpha_2]\!]\rangle.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
[\rho^\theta(X_1+\alpha_1), \rho^\theta(X_2+\alpha_2)] &= [\rho(X_1+\alpha_1), \rho(X_2+\alpha_2)]\\
&\quad + \rho(X_1+\alpha_1)\langle\theta, X_2+\alpha_2\rangle - \rho(X_2+\alpha_2)\langle\theta, X_1+\alpha_1\rangle\\
&= [a(X_1), a(X_2)] + [a_*(\alpha_1), a_*(\alpha_2)] + [a(X_1), a_*(\alpha_2)] - [a(X_2), a_*(\alpha_1)]\\
&\quad + \rho(X_1+\alpha_1)\langle\theta, X_2+\alpha_2\rangle - \rho(X_2+\alpha_2)\langle\theta, X_1+\alpha_1\rangle
\end{aligned}$$

and, by (50)–(52), we conclude that $[\rho^\theta(X_1+\alpha_1), \rho^\theta(X_2+\alpha_2)] = [\rho^\theta(X_1+\alpha_1), \rho^\theta(X_2+\alpha_2)]$.
(iii) For any sections $e_1 = X_1 + \alpha_1$, $e_2 = X_2 + \alpha_2$ and $h = Y + \beta$ of $A \oplus A^*$, we compute

$$\begin{aligned}
\rho^\theta(h)(e_1, e_2)_+ &= \rho(h)(e_1, e_2)_+ + \langle\theta, h\rangle(e_1, e_2)_+\\
&= \tfrac{1}{2}a(Y)(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle) + \tfrac{1}{2}a_*(\beta)(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle) + \langle\theta, h\rangle(e_1, e_2)_+
\end{aligned} \tag{59}$$

and, taking into account that

$$(h, e_1)_+ + (h, e_1)_- = \langle\beta, X_1\rangle \qquad \text{and} \qquad (h, e_1)_+ - (h, e_1)_- = \langle\alpha_1, Y\rangle$$

$$\begin{aligned}
([\![h, e_1]\!] + \mathcal{D}^\theta(h, e_1)_+, e_2)_+ &= \tfrac{1}{2}\Big(\big\langle[\beta, \alpha_1]^W_* + \mathcal{L}^\phi_Y\alpha_1 - \mathcal{L}^\phi_{X_1}\beta + \mathrm{d}^\phi(\langle\beta, X_1\rangle), X_2\big\rangle\\
&\quad + \big\langle\alpha_2, [Y, X_1]^\phi + \mathcal{L}^W_{*\beta}X_1 - \mathcal{L}^W_{*\alpha_1}Y + \mathrm{d}^W_*(\langle\alpha_1, Y\rangle)\big\rangle\Big).
\end{aligned} \tag{60}$$

Similarly,

$$\begin{aligned}
(e_1, [\![h, e_2]\!] + \mathcal{D}^\theta(h, e_2)_+)_+ &= \tfrac{1}{2}\Big(\big\langle[\beta, \alpha_2]^W_* + \mathcal{L}^\phi_Y\alpha_2 - \mathcal{L}^\phi_{X_2}\beta + \mathrm{d}^\phi(\langle\beta, X_2\rangle), X_1\big\rangle\\
&\quad + \big\langle\alpha_1, [Y, X_2]^\phi + \mathcal{L}^W_{*\beta}X_2 - \mathcal{L}^W_{*\alpha_2}Y + \mathrm{d}^W_*(\langle\alpha_2, Y\rangle)\big\rangle\Big).
\end{aligned} \tag{61}$$

Adding up (60) and (61) we obtain, using the equality $\big\langle\mathcal{L}^\phi_{X_1}\beta, X_2\big\rangle = i_{X_2}\big(i_{X_1}\mathrm{d}^\phi\beta + \mathrm{d}^\phi i_{X_1}\beta\big)$ and its dual version,

$$\begin{aligned}
&([\![h, e_1]\!] + \mathcal{D}^\theta(h, e_1)_+, e_2)_+ + (e_1, [\![h, e_2]\!] + \mathcal{D}^\theta(h, e_2)_+)_+\\
&\quad = \tfrac{1}{2}\big\{\mathcal{L}^\phi_Y(\langle\alpha_1, X_2\rangle) - i_{X_2}\big(i_{X_1}\mathrm{d}^\phi\beta\big) + \mathcal{L}^W_{*\beta}(\langle\alpha_2, X_1\rangle) - i_{\alpha_2}\big(i_{\alpha_1}\mathrm{d}^W_*Y\big)\\
&\qquad + \mathcal{L}^\phi_Y(\langle\alpha_2, X_1\rangle) - i_{X_1}\big(i_{X_2}\mathrm{d}^\phi\beta\big) + \mathcal{L}^W_{*\beta}(\langle\alpha_1, X_2\rangle) - i_{\alpha_1}\big(i_{\alpha_2}\mathrm{d}^W_*Y\big)\big\}\\
&\quad = \tfrac{1}{2}\{a(Y)(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle) + \langle\phi, Y\rangle(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle)\\
&\qquad + a_*(\beta)(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle) + \langle\beta, W\rangle(\langle\alpha_1, X_2\rangle + \langle\alpha_2, X_1\rangle)\}\\
&\quad = \rho(h)(e_1, e_2)_+ + \langle\theta, h\rangle(e_1, e_2)_+.
\end{aligned} \tag{62}$$

(iv) For any $f, g \in C^\infty(M, \mathbb{R})$, we have

$$(\mathcal{D}^\theta f, \mathcal{D}^\theta g)_+ = \tfrac{1}{2}(\langle\mathrm{d}^\phi f, \mathrm{d}^W_* g\rangle + \langle\mathrm{d}^\phi g, \mathrm{d}^W_* f\rangle) \overset{(51)}{=} 0.$$

The first example of a generalized Courant algebroid which is a double of a generalized Lie bialgebroid, comes from Jacobi manifolds. As is illustrated by example 2.6, we can associate with each Jacobi manifold (M, Λ, E) a generalized Lie bialgebroid

$$(((T^*M \times \mathbb{R}, [\cdot,\cdot]_{(\Lambda,E)}, \pi \circ (\Lambda, E)^{\#}), (-E, 0)), ((TM \times \mathbb{R}, [\cdot,\cdot], \pi), (0, 1))).$$

Let us denote by $\mathcal{E}^1(M)$ the vector bundle over M, $(TM \times \mathbb{R}) \oplus (T^*M \times \mathbb{R}) \to M$. The next proposition is an immediate consequence of proposition 4.1. □

Proposition 4.3. *If (M, Λ, E) is a Jacobi manifold, then the pair $(\mathcal{E}^1(M), \theta)$, with $\theta = (0, 1) + (-E, 0) \in \Gamma((\mathcal{E}^1(M))^*)$ is a generalized Courant algebroid.*

The next example of a generalized Courant algebroid also appears in [5].

Example 4.4. Let us now consider the Lie algebroid $(TM \times \mathbb{R}, [\cdot,\cdot], \pi)$ and its 1-cocycle $\phi = (0, 1) \in \Gamma(T^*M \times \mathbb{R})$ (see (13)). Its dual vector bundle $T^*M \times \mathbb{R}$ is also a Lie algebroid if we endow the space of sections with an Abelian Lie algebra structure and take the null anchor map; that is $[\cdot,\cdot]_* = 0$ and $\rho_* = 0$. Moreover, the section $W = (0, 0)$ of $TM \times \mathbb{R}$ is obviously a 1-cocycle for the Lie algebroid $T^*M \times \mathbb{R}$ and, from definitions 2.4 or 2.5, it is immediate to see that the pair $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (0, 0)))$ is a generalized Lie bialgebroid. Therefore, by proposition 4.1, we conclude that the pair $(\mathcal{E}^1(M), \psi)$, with $\psi = ((0, 1) + (0, 0)) \in \Gamma((\mathcal{E}^1(M))^*)$, is a generalized Courant algebroid.

The explicit expression of the bracket (45) on the space of sections of the generalized Courant algebroid of the previous example is the following:

$$[\![(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)]\!] = [(X_1, f_1), (X_2, f_2)]^{(0,1)} + \left(\mathcal{L}^{(0,1)}_{(X_1,f_1)}(\alpha_2, g_2) - \mathcal{L}^{(0,1)}_{(X_2,f_2)}(\alpha_1, g_1) + d^{(0,1)}(e_1, e_2)_-\right) \tag{63}$$

with $e_i = (X_i, f_i) + (\alpha_i, g_i)$, $i = 1, 2$, being arbitrary sections of $\mathcal{E}^1(M)$ and

$$(e_1, e_2)_- = \tfrac{1}{2}\{\langle\alpha_1, X_2\rangle - \langle\alpha_2, X_1\rangle + f_2 g_1 - f_1 g_2\} \tag{64}$$

A simple computation gives

$$\begin{aligned}\mathcal{L}^{(0,1)}_{(X_1,f_1)}(\alpha_2, g_2) &= \langle(0, 1), (X_1, f_1)\rangle(\alpha_2, g_2) + i_{(X_1,f_1)}\,\mathrm{d}(\alpha_2, g_2) + \mathrm{d}(i_{(X_1,f_1)}(\alpha_2, g_2)) \\ &= (f_1\alpha_2, f_1 g_2) + i_{(X_1,f_1)}(\delta\alpha_2, -\delta g_2) + \mathrm{d}(i_{X_1}\alpha_2 + f_1 g_2, 0) \\ &= \left(f_1\alpha_2 + \mathcal{L}_{X_1}\alpha_2 + g_2\delta f_1, f_1 g_2 + X_1(g_2)\right)\end{aligned} \tag{65}$$

and, analogously,

$$\mathcal{L}^{(0,1)}_{(X_2,f_2)}(\alpha_1, g_1) = (f_2\alpha_1 + \mathcal{L}_{X_2}\alpha_1 + g_1\delta f_2, f_2 g_1 + X_2(g_1)). \tag{66}$$

We also compute

$$\begin{aligned}\mathrm{d}^{(0,1)}(e_1, e_2)_- &= \mathrm{d}(e_1, e_2)_- + (0, (e_1, e_2)_-) \\ &= \left(\tfrac{1}{2}\delta(\langle\alpha_1, X_2\rangle - \langle\alpha_2, X_1\rangle + f_2 g_1 - f_1 g_2), \tfrac{1}{2}(\langle\alpha_1, X_2\rangle \right. \\ &\quad \left. - \langle\alpha_2, X_1\rangle + f_2 g_1 - f_1 g_2)\right).\end{aligned} \tag{67}$$

So, from (65), (66) and (67), we obtain

$$\begin{aligned}[\![(X_1, f_1) + (\alpha_1, g_1), (X_2, f_2) + (\alpha_2, g_2)]\!] &= ([X_1, X_2], X_1(f_2) - X_2(f_1)) \\ &\quad + \left(f_1\alpha_2 - f_2\alpha_1 + \mathcal{L}_{X_1}\alpha_2 - \mathcal{L}_{X_2}\alpha_1 + \tfrac{1}{2}\delta(\langle\alpha_1, X_2\rangle - \langle\alpha_2, X_1\rangle)\right. \\ &\quad + \tfrac{1}{2}(f_2\delta g_1 - g_1\delta f_2 + g_2\delta f_1 - f_1\delta g_2), X_1(g_2) - X_2(g_1) \\ &\quad \left. + \tfrac{1}{2}(\langle\alpha_1, X_2\rangle - \langle\alpha_2, X_1\rangle) + \tfrac{1}{2}(f_1 g_2 - f_2 g_1)\right).\end{aligned} \tag{68}$$

This is exactly the bracket introduced in the space of sections of $\mathcal{E}^1(M)$ by Wade in [26]. If $L \subset \mathcal{E}^1(M)$ is a Dirac structure for the generalized Courant algebroid $(\mathcal{E}^1(M), (0,1)+(0,0))$, then by proposition 3.9, $(L, \rho|_L, [\cdot,\cdot]|_L)$ is a Lie algebroid over M; and this is the content of theorem 3.4 in [26]. Finally, also by proposition 3.9, we have that $(0,1)+(0,0) \in \Gamma(L^*)$ is a 1-cocycle for the Lie algebroid $(L, \rho|_L, [\cdot,\cdot]|_L)$. This fact was pointed out in [7].

At this point we can make the following analogy: likewise the bracket introduced by Courant in [1] is obtained from the bracket in the double $TM \oplus T^*M$ of the Lie bialgebroid (TM, T^*M) ([19]) in the particular case where T^*M is endowed with the *null* Lie algebroid structure, the bracket introduced by Wade in [26] is obtained from the bracket (45) in the double $\mathcal{E}^1(M)$ of the generalized Lie bialgebroid $((TM \times \mathbb{R}, (0,1)), (T^*M \times \mathbb{R}, (0,0)))$, in the particular case where $T^*M \times \mathbb{R}$ is endowed with the *null* Lie algebroid structure.

5. Dirac structures for generalized Courant algebroids

In this section, we present some examples of Dirac structures for generalized Courant algebroids which are graphs of bivector fields.

Let $((A,\phi),(A^*,W))$ be a generalized Lie bialgebroid over M and Ω a A-bivector field, i.e. $\Omega \in \Gamma(\bigwedge^2 A)$. Let us denote by $\Omega^\#$ the associated vector bundle map, $\Omega^\# : A^* \to A$, and by L the graph of $\Omega^\#$, considered as a sub-bundle of $A \oplus A^*$,

$$L = \{\Omega^\#\alpha + \alpha, \alpha \in A^*\}.$$

Proposition 5.1. *The graph of* $\Omega^\#$ *is a Dirac structure for the generalized Courant algebroid* $(A \oplus A^*, \phi + W)$ *if and only if the Maurer–Cartan type equation*

$$\mathrm{d}_*^W \Omega + \tfrac{1}{2}[\Omega,\Omega]^\phi = 0 \tag{69}$$

holds.

Proof. If $\Omega^\#\alpha + \alpha, \Omega^\#\beta + \beta \in L = \text{graph } \Omega^\#$ then, since Ω is skew-symmetric, we have

$$(\Omega^\#\alpha + \alpha, \Omega^\#\beta + \beta)_+ = \tfrac{1}{2}(\langle\alpha, \Omega^\#\beta\rangle + \langle\beta, \Omega^\#\alpha\rangle) = 0 \tag{70}$$

and L is a maximal isotropic sub-bundle of $A \oplus A^*$.

It remains to show that L is integrable. Or the bracket (45) is expressed, in this case, for any sections $\Omega^\#\alpha + \alpha$ and $\Omega^\#\beta + \beta$ of L, as follows:

$$\begin{aligned}[\![\Omega^\#\alpha + \alpha, \Omega^\#\beta + \beta]\!] &= \big([\Omega^\#\alpha, \Omega^\#\beta] + \mathcal{L}^W_{*\alpha}(\Omega^\#\beta) - \mathcal{L}^W_{*\beta}(\Omega^\#\alpha) + \mathrm{d}^W_*(\Omega(\alpha,\beta))\big) \\ &\quad + \big([\alpha,\beta]_* + \mathcal{L}^\phi_{\Omega^\#\alpha}\beta - \mathcal{L}^\phi_{\Omega^\#\beta}\alpha - \mathrm{d}^\phi(\Omega(\alpha,\beta))\big).\end{aligned} \tag{71}$$

If we denote by $[\alpha,\beta]_\Omega$ the last three terms of (71),

$$[\alpha,\beta]_\Omega = \mathcal{L}^\phi_{\Omega^\#\alpha}\beta - \mathcal{L}^\phi_{\Omega^\#\beta}\alpha - \mathrm{d}^\phi(\Omega(\alpha,\beta)), \tag{72}$$

then, L is integrable if and only if

$$[\Omega^\#\alpha, \Omega^\#\beta] + \mathcal{L}^W_{*\alpha}(\Omega^\#\beta) - \mathcal{L}^W_{*\beta}(\Omega^\#\alpha) + \mathrm{d}^W_*(\Omega(\alpha,\beta)) = \Omega^\#([\alpha,\beta]_* + [\alpha,\beta]_\Omega). \tag{73}$$

For any $\alpha, \beta \in \Gamma(A^*)$, a straightforward computation leads to

$$(\mathrm{d}^W_*\Omega)(\alpha,\beta) = \mathcal{L}^W_{*\alpha}(\Omega^\#\beta) - \mathcal{L}^W_{*\beta}(\Omega^\#\alpha) + \mathrm{d}^W_*(\Omega(\alpha,\beta)) - \Omega^\#([\alpha,\beta]_*) \tag{74}$$

and so (73) is equivalent to

$$[\Omega^\#\alpha, \Omega^\#\beta] + (\mathrm{d}^W_*\Omega)(\alpha,\beta) = \Omega^\#([\alpha,\beta]_\Omega). \tag{75}$$

On the other hand, for any A-bivector field Ω, the following formula holds (see [13]):

$$[\Omega^\#\alpha, \Omega^\#\beta] = \Omega^\#\big(\mathcal{L}_{\Omega^\#\alpha}\beta - \mathcal{L}_{\Omega^\#\beta}\alpha - \mathrm{d}(\Omega(\alpha,\beta))\big) + \tfrac{1}{2}[\Omega,\Omega](\alpha,\beta) \tag{76}$$

and from (8),

$$\tfrac{1}{2}[\Omega,\Omega]^{\phi}(\alpha,\beta)=\tfrac{1}{2}[\Omega,\Omega](\alpha,\beta)+(\Omega^{\#}\phi\wedge\Omega)(\alpha,\beta). \tag{77}$$

Using (76) and (77) in (75), we conclude that L is integrable if and only if

$$\begin{aligned}\Omega^{\#}\big(\mathcal{L}_{\Omega^{\#}\alpha}\beta-\mathcal{L}_{\Omega^{\#}\beta}\alpha-\mathrm{d}(\Omega(\alpha,\beta))\big)+\tfrac{1}{2}[\Omega,\Omega]^{\phi}(\alpha,\beta)&\\ -((\Omega^{\#}\phi)\wedge\Omega)(\alpha,\beta)+(\mathrm{d}_*^{W}\Omega)(\alpha,\beta)=\Omega^{\#}([\alpha,\beta]_{\Omega})&\end{aligned} \tag{78}$$

that is, if and only if,

$$\begin{aligned}(\mathrm{d}_*^{W}\Omega)(\alpha,\beta)+\tfrac{1}{2}[\Omega,\Omega]^{\phi}(\alpha,\beta)&\\ =\Omega^{\#}([\alpha,\beta]_{\Omega})-\underbrace{\Omega^{\#}(\mathcal{L}_{\Omega^{\#}\alpha}\beta-\mathcal{L}_{\Omega^{\#}\beta}\alpha-d(\Omega(\alpha,\beta)))+((\Omega^{\#}\phi)\wedge\Omega)(\alpha,\beta)}_{=\Omega^{\#}([\alpha,\beta]_{\Omega})}&\\ =0.&\end{aligned} \tag{79}$$

□

Let us now consider the generalized Courant algebroid $(\mathcal{E}^1(M),\psi)$, with $\psi=(0,1)+(0,0))$, treated in example 4.4. A section of $\bigwedge^2(TM\times\mathbb{R})$ is, in this case, a pair (Λ,E) where Λ and E are, respectively, a bivector field and a vector field on M. The graph of $(\Lambda,E)^{\#}$ is a sub-bundle L of $\mathcal{E}^1(M)$ whose space of sections is

$$\begin{aligned}\Gamma(L)&=\{(\Lambda,E)^{\#}(\alpha,g)+(\alpha,g),(\alpha,g)\in\Omega^1(M)\times C^{\infty}(M,\mathbb{R})\}\\ &\overset{(2)}{=}\{(\Lambda^{\#}\alpha+gE,-\langle\alpha,E\rangle)+(\alpha,g),(\alpha,g)\in\Omega^1(M)\times C^{\infty}(M,\mathbb{R})\}.\end{aligned} \tag{80}$$

By proposition 5.1, L is a Dirac structure for the generalized Courant algebroid $(\mathcal{E}^1(M),(0,1)+(0,0))$ if and only if

$$\underbrace{\mathrm{d}_*^{(0,0)}(\Lambda,E)}_{=0}+\tfrac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}=0\quad\Leftrightarrow\quad[(\Lambda,E),(\Lambda,E)]^{(0,1)}=0.$$

But $[(\Lambda,E),(\Lambda,E)]^{(0,1)}=0$ *if and only if* (M,Λ,E) is a Jacobi manifold (see [6]). So, we obtain a characterization of Jacobi manifolds in terms of Dirac structures and we recover a result from [26]: *the graph of* (Λ,E) *is a Dirac structure for* $(\mathcal{E}^1(M),\psi)$ *if and only if* (Λ,E) *is a Jacobi structure on M*.

We recall that two Jacobi structures (Λ,E) and (Λ',E') on a manifold M are said to be *compatible* if their sum is still a Jacobi structure on M [21].

Proposition 5.2. *Let* (M,Λ,E) *be a Jacobi manifold and* $(\mathcal{E}^1(M),\theta)$*, with* $\theta=(0,1)+(-E,0)\in\Gamma((\mathcal{E}^1(M))^*)$ *the generalized Courant algebroid associated, as in proposition 4.3, and let* (Λ',E') *be a section of* $\bigwedge^2(TM\times\mathbb{R})$*. Then, the pair* $(\Lambda+\Lambda',E+E')=(\Lambda,E)+(\Lambda',E')$ *determines a Jacobi structure on M if and only if the graph of* (Λ',E') *is a Dirac structure for* $(\mathcal{E}^1(M),\theta)$*. Moreover,* (Λ',E') *is a Jacobi structure on M, compatible with* (Λ,E)*, if and only if*

$$\mathrm{d}_*^{(-E,0)}(\Lambda',E')=0\qquad\text{and}\qquad[(\Lambda',E'),(\Lambda',E')]^{(0,1)}=0.$$

Proof. As we have already remarked, $(\Lambda,E)+(\Lambda',E')$ determines a Jacobi structure on M if and only if

$$[(\Lambda,E)+(\Lambda',E'),(\Lambda,E)+(\Lambda',E')]^{(0,1)}=0$$

or equivalently, if and only if

$$2[(\Lambda,E),(\Lambda',E')]^{(0,1)}+[(\Lambda',E'),(\Lambda',E')]^{(0,1)}=0. \tag{81}$$

But, since $d_*^{(-E,0)}(\Lambda', E') = [(\Lambda, E), (\Lambda', E')]^{(0,1)}$ (see [6]), equation (81) turns to

$$d_*^{(-E,0)}(\Lambda', E') + \frac{1}{2}[(\Lambda', E'), (\Lambda', E')]^{(0,1)} = 0$$

which means, by proposition 5.1, that the graph of (Λ', E') is a Dirac structure for $(\mathcal{E}^1(M), \theta)$.

For the last assertion, we have that (Λ', E') is a Jacobi structure on M if and only if $[(\Lambda', E'), (\Lambda', E')]^{(0,1)} = 0$. Moreover,

$$d_*^{(-E,0)}(\Lambda', E') = 0 \quad \Longleftrightarrow \quad [\Lambda, \Lambda'] + E \wedge \Lambda' + E' \wedge \Lambda = 0 \qquad [E', \Lambda] + [E, \Lambda'] = 0$$

and these are the conditions that assure the compatibility of the Jacobi structures (Λ, E) and (Λ', E') (see [21]). □

6. Null Dirac structures and characteristic pairs

Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid and $D \subset A$ a sub-bundle of A. We denote by $D^\perp \subset A^*$ the conormal bundle of D,

$$D^\perp = \{\alpha \in A^* : \langle \alpha, X \rangle = 0, \forall X \in D\}.$$

Proposition 6.1. *The sub-bundle $L = D \oplus D^\perp$ of $A \oplus A^*$ is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if D and $D^\perp$ are Lie subalgebroids of A and A^*, respectively. In this case, L is said to be a null Dirac structure.*

Proof. For any $X_1 + \alpha_1, X_2 + \alpha_2 \in L = D \oplus D^\perp$,

$$(X_1 + \alpha_1, X_2 + \alpha_2)_\pm = \frac{1}{2}(\langle \alpha_1, X_2 \rangle \pm \langle \alpha_2, X_1 \rangle) = 0$$

and L is a maximal isotropic sub-bundle of $A \oplus A^*$.

If L is a Dirac structure for $(A \oplus A^*, \phi + W)$, then L is integrable, i.e. L is closed with respect to the bracket $[\![\cdot, \cdot]\!]$ given by (45).

For any sections X_1 and X_2 of $D \subset L$, we compute

$$[\![X_1 + 0, X_2 + 0]\!] = [X_1, X_2] + 0 \in \Gamma(L).$$

Therefore, $[X_1, X_2] \in \Gamma(D)$ and D is a Lie subalgebroid of A ([18]). An analogous reasoning shows that $D^\perp$ is a Lie subalgebroid of A^*.

Conversely, let us suppose that D and $D^\perp$ are Lie subalgebroids of A and A^*, respectively. Since we have, for any sections $X_1 + \alpha_1, X_2 + \alpha_2$ of L,

$$[\![X_1 + \alpha_1, X_2 + \alpha_2]\!] = \left([X_1, X_2] + \mathcal{L}^W_{*\alpha_1} X_2 - \mathcal{L}^W_{*\alpha_2} X_1\right) + \left([\alpha_1, \alpha_2]_* + \mathcal{L}^\phi_{X_1} \alpha_2 - \mathcal{L}^\phi_{X_2} \alpha_1\right) \qquad (82)$$

for concluding that L is integrable, we only have to verify that $\mathcal{L}^W_{*\alpha_1} X_2$ and $\mathcal{L}^W_{*\alpha_2} X_1$ (resp. $\mathcal{L}^\phi_{X_1} \alpha_2$ and $\mathcal{L}^\phi_{X_2} \alpha_1$) are sections of D (resp. $D^\perp$). Or, if β is a section of $D^\perp$,

$$\begin{aligned}\left\langle \beta, \mathcal{L}^W_{*\alpha_1} X_2 \right\rangle &= \left\langle \beta, \mathcal{L}_{*\alpha_1} X_2 \right\rangle + \langle \alpha_1, W \rangle \underbrace{\langle \beta, X_2 \rangle}_{=0} \\ &= a_*^\perp(\alpha_1)(\langle \beta, X_2 \rangle) - \langle [\beta, \alpha_1]_*, X_2 \rangle \\ &= 0\end{aligned}$$

where $a_*^\perp$ stands for the anchor of the Lie algebroid $D^\perp$. Therefore, $\mathcal{L}^W_{*\alpha_1} X_2 \in \Gamma(D)$ and, in the same way, one has $\mathcal{L}^W_{*\alpha_2} X_1 \in \Gamma(D)$. Similarly, one can show that $\mathcal{L}^\phi_{X_1} \alpha_2$ and $\mathcal{L}^\phi_{X_2} \alpha_1$ are sections of $D^\perp$. □

Remark 6.2. In [6], the following result was obtained. Let $(\mathcal{G}, [\cdot, \cdot])$ be a Lie algebra and $\mathcal{Z}(\mathcal{G})$ the centre of $\mathcal{G}$. If $r \in \bigwedge^2 \mathcal{G}$, $X_0 \in \mathcal{Z}(\mathcal{G})$ and $[r, r] - 2X_0 \wedge r = 0$, then the pair

$((\mathcal{G}, 0), (\mathcal{G}^*, X_0))$ is a generalized Lie bialgebra, i.e., a generalized Lie bialgebroid over a point. The Lie bracket on $\mathcal{G}^*$ is given by

$$[\alpha, \beta]_* = \mathrm{ad}^*_{r^\#(\alpha)}\beta - \mathrm{ad}^*_{r^\#(\beta)}\alpha - i_{X_0}(\alpha \wedge \beta) \tag{83}$$

where $\mathrm{ad}^* : \mathcal{G} \times \mathcal{G}^* \to \mathcal{G}^*$ is the coadjoint representation of $\mathcal{G}$ on $\mathcal{G}^*$ defined by $(\mathrm{ad}^*_X\alpha)(Y) = -\alpha([X, Y])$, for $X, Y \in \mathcal{G}$ and $\alpha \in \mathcal{G}^*$. By proposition 4.1 $(\mathcal{G} \oplus \mathcal{G}^*, 0 + X_0)$ is a generalized Courant algebroid. We will use this approach in the next example.

Example 6.3. Let $\mathcal{G} = gl(2, \mathbb{R})$ be the Lie algebra of the general linear Lie group $GL(2, \mathbb{R})$ and let

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

be a basis of $\mathcal{G}$. The Lie bracket on $\mathcal{G}$ is defined by

$$[e_1, e_2] = e_3 \qquad [e_1, e_3] = -2e_1 \qquad [e_2, e_3] = 2e_2 \qquad e_4 \in \mathcal{Z}(\mathcal{G}).$$

If we take $r = e_1 \wedge e_3 + \left(e_1 - \frac{1}{2}e_3\right) \wedge e_4$ and $X_0 = e_4$, then $[r, r] - 2X_0 \wedge r = 0$ holds. So the pair $((\mathcal{G}, 0), (\mathcal{G}^*, e_4))$ is a generalized Lie bialgebra (see [6]) and its double $(\mathcal{G} \oplus \mathcal{G}^*, e_4)$ is a generalized Courant algebroid (over a point).

Let $D \subset \mathcal{G}$ be the vector space generated by the elements e_1 and e_3 of $\mathcal{G}$. D is a Lie subalgebra of $\mathcal{G}$. Let $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ be a basis of $\mathcal{G}^*$, dual of the basis $\{e_1, e_2, e_3, e_4\}$ of $\mathcal{G}$ and consider the pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{G}$ and $\mathcal{G}^*$ given by

$$\langle X, A \rangle = \mathrm{Tr}(X^T A) \qquad X \in \mathcal{G} \qquad A \in \mathcal{G}^*.$$

Then $D^\perp \subset \mathcal{G}^*$ is generated by the elements e_2^* and e_4^* of $\mathcal{G}^*$.

Given $A, B \in D^\perp$, let us see that $[A, B]_* \in D^\perp$, where $[\cdot, \cdot]_*$ is the bracket (83) on $\mathcal{G}^*$,

$$[A, B]_* = \mathrm{ad}^*_{r^\#(A)}B - \mathrm{ad}^*_{r^\#(B)}A - i_{X_0}(A \wedge B).$$

If $Y \in D$, then

$$\left(\mathrm{ad}^*_{r^\#(A)}B\right)(Y) = -\langle B, [r^\#(A), Y] \rangle = 0. \tag{84}$$

In fact, with $A \in D^\perp$, one has $r^\#(A) \in D$ and, since D is closed, $[r^\#(A), Y] \in D$, which implies, from (84), that $\mathrm{ad}^*_{r^\#(A)}B \in D^\perp$. Analogously, $\mathrm{ad}^*_{r^\#(B)}A \in D^\perp$. Finally,

$$i_{X_0}(A \wedge B) = (\mathrm{Tr}(A^T X_0))B - (\mathrm{Tr}(B^T X_0))A$$

so $i_{X_0}(A \wedge B) \in D^\perp$ and we conclude that $[A, B]_* \in D^\perp$. By proposition 6.1, $L = D \oplus D^\perp \subset gl(2, \mathbb{R}) \oplus gl^*(2, \mathbb{R})$ is a null Dirac structure for the generalized Courant algebroid $(gl(2, \mathbb{R}) \oplus gl^*(2, \mathbb{R}), e_4)$.

Let us now recall the notion of *characteristic pair*, introduced in [16], which provides Dirac structures generalizing both the case of a graph of a bivector field (treated in the previous section) and that of a null Dirac structure.

Let A be a vector bundle, $D \subset A$ a sub-bundle of A and Ω a bivector field, $\Omega \in \Gamma(\bigwedge^2 A)$. Consider the sub-bundle L of $A \oplus A^*$, given by

$$L = \{X + \Omega^\#\alpha + \alpha, X \in D, \alpha \in D^\perp\} = D \oplus \mathrm{graph}(\Omega^\#|_{D^\perp}). \tag{85}$$

Clearly $L \subset A \oplus A^*$ is maximally isotropic with respect to the symmetric bilinear form (43). In what follows, we will assume that $D = L \cap A$ is of constant rank.

Definition 6.4 ([16]). *The pair (D, Ω) is called the characteristic pair of the sub-bundle L of $A \oplus A^*$ given by (85), while $D = L \cap A$ is called the characteristic sub-bundle of L.*

As is remarked in [16], the restricted bundle map $\Omega^{\#}|_{D^{\perp}}$ is equivalent to a bivector field on the quotient bundle A/D. Thus, two characteristic pairs (D_1, Ω_1) and (D_2, Ω_2) determine the same sub-bundle L defined by (85) if and only if

$$D_1 = D_2 = D \qquad \text{and} \qquad \Omega_1 - \Omega_2 = 0\,(\text{mod}\, D)$$

where by $\Omega_1 - \Omega_2 = 0\,(\text{mod}\, D)$ we mean $\left(\Omega_1^{\#}\alpha - \Omega_2^{\#}\alpha\right) \in D, \forall \alpha \in D^{\perp}$.

We are interested in characteristic pairs for the case where $((A, \phi), (A^*, W))$ is a generalized Lie bialgebroid.

Theorem 6.5. *Let $((A, \phi), (A^*, W))$ be a generalized Lie bialgebroid and $L \subset A \oplus A^*$ a maximal isotropic sub-bundle of $A \oplus A^*$ defined by a characterisitc pair (D, Ω), i.e*

$$L = \{X + \Omega^{\#}\alpha + \alpha,\ X \in D, \alpha \in D^{\perp}\} = D \oplus \text{graph}(\Omega^{\#}|_{D^{\perp}}).$$

Then L is a Dirac structure for the generalized Courant algebroid $(A \oplus A^*, \phi + W)$ if and only if

(i) D is a Lie subalgebroid of A;
(ii) $\mathrm{d}_*^W\Omega + \frac{1}{2}[\Omega, \Omega]^{\phi} = 0\,(\text{mod}\, D)$;
(iii) for any $\alpha, \beta \in \Gamma(D^{\perp})$, $[\alpha, \beta]_* + [\alpha, \beta]_{\Omega} \in \Gamma(D^{\perp})$, where $[\cdot, \cdot]_{\Omega}$ is the bracket (72).

Proof. We only have to verify that the closeness of L is equivalent to conditions (i), (ii) and (iii).

If $X + \Omega^{\#}\alpha + \alpha$ and $Y + \Omega^{\#}\beta + \beta$ are any sections of $L = D \oplus \text{graph}(\Omega^{\#}|_{D^{\perp}})$, then

$$[\![X + \Omega^{\#}\alpha + \alpha, Y + \Omega^{\#}\beta + \beta]\!] = [\![X, Y]\!] + [\![X, \Omega^{\#}\beta + \beta]\!] + [\![\Omega^{\#}\alpha + \alpha, Y]\!] + [\![\Omega^{\#}\alpha + \alpha, \Omega^{\#}\beta + \beta]\!]. \tag{86}$$

Regarding the first term of the second member of equation (86), $[\![X, Y]\!] = [X, Y] \in \Gamma(D)$ *if and only if* D is a Lie subalgebroid of A (condition (i)). The second and third terms of second member of (86) are of the same type.

Or,

$$\begin{aligned}[\![X, \Omega^{\#}\beta + \beta]\!] &\overset{(45)}{=} \left([X, \Omega^{\#}\beta] - \mathcal{L}_{*\beta}^{W}X\right) + \mathcal{L}_X^{\phi}\beta \\ &= \left([X, \Omega^{\#}\beta] - \mathcal{L}_{*\beta}^{W}X - \Omega^{\#}\left(\mathcal{L}_X^{\phi}\beta\right)\right) + \Omega^{\#}\left(\mathcal{L}_X^{\phi}\beta\right) + \mathcal{L}_X^{\phi}\beta. \end{aligned} \tag{87}$$

Moreover, for any $Z \in \Gamma(D)$,

$$\begin{aligned}\left\langle \mathcal{L}_X^{\phi}\beta, Z\right\rangle &= \langle \mathcal{L}_X\beta, Z\rangle + \langle \phi, X\rangle \underbrace{\langle \beta, Z\rangle}_{=0} \\ &= a(X)(\langle \beta, Z\rangle) - \langle \beta, [X, Z]\rangle \\ &= -\langle \beta, [X, Z]\rangle\end{aligned}$$

and so, $\mathcal{L}_X^{\phi}\beta \in \Gamma(D^{\perp})$ *if and only if* D is a Lie subalgebroid of A (condition (i)).

In this case, from (87) we deduce that

$$[\![X, \Omega^{\#}\beta + \beta]\!] \in \Gamma(L) \quad \text{if and only if} \quad [X, \Omega^{\#}\beta] - \mathcal{L}_{*\beta}^{W}X - \Omega^{\#}\left(\mathcal{L}_X^{\phi}\beta\right) \in \Gamma(D). \tag{88}$$

With $\alpha \in \Gamma(D^{\perp})$, we compute

$$\begin{aligned}&\left\langle \alpha, [X, \Omega^{\#}\beta] - \mathcal{L}_{*\beta}^{W}X - \Omega^{\#}\left(\mathcal{L}_X^{\phi}\beta\right)\right\rangle \\ &\quad = -\langle \alpha, [\Omega^{\#}\beta, X]\rangle - a_*(\beta)(\langle \alpha, X\rangle) + \langle [\beta, \alpha]_*, X\rangle + a(X)(\langle \beta, \Omega^{\#}\alpha\rangle) \\ &\qquad - \langle \beta, [X, \Omega^{\#}\alpha]\rangle + \langle \phi, X\rangle\Omega(\alpha, \beta) \\ &\quad = \langle [\beta, \alpha]_*, X\rangle - \langle \alpha, [\Omega^{\#}\beta, X]\rangle - \langle \beta, [X, \Omega^{\#}\alpha]\rangle + \langle \mathrm{d}^{\phi}(\Omega(\alpha, \beta)), X\rangle \\ &\quad = \langle [\beta, \alpha]_* + \mathcal{L}_{\Omega^{\#}\beta}\alpha - \mathcal{L}_{\Omega^{\#}\alpha}\beta + \mathrm{d}^{\phi}(\Omega(\alpha, \beta)), X\rangle \\ &\quad = -\langle [\alpha, \beta]_* + [\alpha, \beta]_{\Omega}, X\rangle. \end{aligned} \tag{89}$$

So (88) is equivalent to

$$[\![X, \Omega^\# \beta + \beta]\!] \in \Gamma(L) \quad \text{if and only if} \quad [\alpha, \beta]_* + [\alpha, \beta]_\Omega \in \Gamma(D^\perp) \tag{90}$$

(condition (iii)).

For the last term of the second member of equation (86), a straightforward calculation, using (72), (74), (76) and (77), gives

$$\begin{aligned}[\![\Omega^\#\alpha + \alpha, \Omega^\#\beta + \beta]\!] &= \left(\mathrm{d}_*^W \Omega + \tfrac{1}{2}[\Omega, \Omega]^\phi\right)(\alpha, \beta) \\ &\quad + \Omega^\#([\alpha, \beta]_* + [\alpha, \beta]_\Omega) + ([\alpha, \beta]_* + [\alpha, \beta]_\Omega).\end{aligned} \tag{91}$$

So, we conclude that the bracket (86) is a section of L *if and only if* the conditions (i), (ii) and (iii) hold. □

Example 6.6. Let $\mathcal{G} = \mathbb{R}^2 \times (\mathbb{R}^2)^* \times \mathbb{R} \cong \mathbb{R}^5$ be the Heisenberg Lie algebra, endowed with the Lie bracket defined, for $i, j = 1, 2$, by

$$[e_i, e_j] = [e^i, e^j] = [e_i, h] = [e^i, h] = 0 \qquad [e_i, e^j] = \delta_{ij} h$$

where $\{e_1, e_2, e^1, e^2, h\}$ is a basis of $\mathcal{G}$. With $r \in \bigwedge^2 \mathcal{G}$ and $X_0 \in \mathcal{G}$ given by

$$r = e_1 \wedge e^1 + e_2 \wedge e^2 \qquad \text{and} \qquad X_0 = h$$

the equation $[r, r] - 2X_0 \wedge r = 0$ holds and $((\mathcal{G}, 0), (\mathcal{G}^*, h))$ is a generalized Lie bialgebra (cf remark 6.2), where the Lie bracket on $\mathcal{G}^*$ is given by (83).

Let D be the vector space generated by the elements e_1, e^1 and h of $\mathcal{G}$; D is closed with respect to the Lie bracket $[\cdot, \cdot]$ on $\mathcal{G}$. Let us denote by $\{f^1, f^2, f_1, f_2, h^*\}$ the basis of $\mathcal{G}^* \cong (\mathbb{R}^5)^*$, dual of $\{e_1, e_2, e^1, e^2, h\}$. If we consider the pairing between $\mathcal{G}$ and $\mathcal{G}^*$ as the usual inner product, then $D^\perp \subset \mathcal{G}^*$ is generated by the elements f^2 and f_2 of $\mathcal{G}^*$. Moreover,

$$r^\#(f^i) = e^i \qquad r^\#(f_i) = -e_i \qquad r^\#(h^*) = 0$$

which implies that $\mathrm{Im}(r^\#|_{D^\perp}) \subset \mathrm{span}\{e_2, e^2\}$. Let

$$L = \{X + r^\#(a) + a, X \in D, a \in D^\perp\} \subset D \oplus \mathrm{graph}(r^\#|_{D^\perp}).$$

By theorem 6.5, L is a Dirac structure for the generalized Courant algebroid $(\mathcal{G} \oplus \mathcal{G}^*, 0 + h)$:

(i) D is a Lie subalgebra of $\mathcal{G}$;

(ii) Since $[r, r] - 2X_0 \wedge r = 0$ holds, it remains to see that $(\mathrm{d}_* r)(a, b, c) = 0$, for $a, b, c \in D^\perp$. Or, if $a, b, c \in D^\perp$, $\left(\mathrm{ad}^*_{r^\#(a)} b\right)(r^\#(c)) = -\langle b, [r^\#(a), r^\#(c)]\rangle = 0$ (because $[r^\#(a), r^\#(c)] \in \mathrm{span}\{h\}$) and $i_{X_0}(a \wedge b)(r^\#(c)) = 0$. So $r([a, b]_*, c) = 0$, and analogously, $r([a, c]_*, b) = r([b, c]_*, a) = 0$, which gives $\mathrm{d}_* r = 0 \pmod D$.

(iii) Let $a, b \in D^\perp$ and $X \in D$. Then, $\langle [a, b]_* + [a, b]_r, X\rangle = 2\left(\mathrm{ad}^*_{r^\#(a)} b\right)(X) - 2\left(\mathrm{ad}^*_{r^\#(b)} a\right)(X) - i_{X_0}(a \wedge b)(X) - \underbrace{\mathrm{d}(r(a, b))}_{=0}(X) = 0$, and so, $[a, b]_* + [a, b]_r \in D^\perp$.

7. Triangular generalized Lie bialgebroids and Dirac structures

In this section, we present a version of theorem 6.5 for the case of a triangular generalized Lie bialgebroid. First, let us recall some results from [6].

Theorem 7.1 ([6])**.** *Let $(A, [\cdot, \cdot], a)$ be a Lie algebroid over M, $\phi \in \Gamma(A^*)$ a 1-cocycle and $P \in \Gamma(\bigwedge^2 A)$ a bivector field such that $[P, P]^\phi = 0$. Then,*

(i) $(A^*, [\cdot, \cdot]_P, a \circ P^\#)$ is a Lie algebroid over M, where $[\cdot, \cdot]_P$ is the bracket (72) associated with P;

(ii) $W = -P^{\#}(\phi) \in \Gamma(A)$ *is a 1-cocycle;*
(iii) *the pair* $((A, \phi), (A^*, W))$ *is a generalized Lie bialgebroid.*

Definition 7.2 ([6]). *A generalized Lie bialgebroid* $((A, \phi), (A^*, W))$ *is said to be a triangular generalized Lie bialgebroid if there exists* $P \in \Gamma(\bigwedge^2 A)$ *such that* $[P, P]^{\phi} = 0$*, the Lie bracket on* $\Gamma(A^*)$ *is* $[\cdot, \cdot]_P$*, the anchor on* A^* *is* $(a \circ P^{\#})$ *and the 1-cocycle W is given by* $W = -P^{\#}(\phi)$.

We will denote by $((A, \phi), (A^*, W), P)$ *a triangular generalized Lie bialgebroid.*

Example 7.3 ([6]). Let (M, Λ, E) be a Jacobi manifold. Then $((TM \times \mathbb{R}, (0, 1)), (T^*M \times \mathbb{R}, (-E, 0)), (\Lambda, E))$ is a triangular generalized Lie bialgebroid. In fact,

- $[(\Lambda, E), (\Lambda, E)]^{(0,1)} = 0$, because (M, Λ, E) is a Jacobi manifold;
- for all sections (α, f) and (β, g) of $T^*M \times \mathbb{R}$,
$$\begin{aligned}[(\alpha, f), (\beta, h)]_{(\Lambda,E)} = {} & \mathcal{L}^{(0,1)}_{(\Lambda,E)^{\#}(\alpha,f)}(\beta, g) - \mathcal{L}^{(0,1)}_{(\Lambda,E)^{\#}(\beta,g)}(\alpha, f) \\ & - \mathrm{d}^{(0,1)}((\Lambda, E)((\alpha, f), (\beta, g)))\end{aligned} \tag{92}$$
- the anchor is $\pi \circ (\Lambda, E)^{\#}$;
- $-(\Lambda, E)^{\#}(0, 1) \overset{(2)}{=} (-E, 0) = W$.

Proposition 7.4. *Let* $((A, \phi), (A^*, W), P)$ *be a triangular generalized Lie bialgebroid and* $L \subset A \oplus A^*$ *a maximal isotropic sub-bundle of* $A \oplus A^*$ *defined by a characteristic pair* (D, Ω)*, i.e.* $L = D \oplus \mathrm{graph}(\Omega^{\#}|_{D^{\perp}})$*. Then L is a Dirac structure for the generalized Courant algebroid* $(A \oplus A^*, \phi + W)$ *if and only if*

(1) D is a Lie subalgebroid of A;
(2) $[P + \Omega, P + \Omega]^{\phi} = 0 \,(\mathrm{mod}\, D)$*;*
(3) for any $Y \in \Gamma(D)$, $\mathcal{L}^{\phi}_Y(P + \Omega) = 0 \,(\mathrm{mod}\, D)$.

Proof. We will show that conditions (2) and (3) are equivalent to (ii) and (iii) of theorem 6.5, respectively. For any bivector field $\Omega \in \Gamma(\bigwedge^2 A)$, we have $\mathrm{d}^W_* \Omega = [P, \Omega]^{\phi}$. Moreover,

$$[P + \Omega, P + \Omega]^{\phi} = 2[P, \Omega]^{\phi} + [\Omega, \Omega]^{\phi}$$

and so we have

$$\begin{aligned}\mathrm{d}^W_* \Omega + \tfrac{1}{2}[\Omega, \Omega]^{\phi} &= [P, \Omega]^{\phi} + \tfrac{1}{2}[\Omega, \Omega]^{\phi} \\ &= \tfrac{1}{2}[P + \Omega, P + \Omega]^{\phi}\end{aligned} \tag{93}$$

which proves the equivalence of (2) and (ii) of theorem 6.5.

On the other hand, for any sections α and β of A^*, it is immediate to verify that

$$[\alpha, \beta]_{P+\Omega} = [\alpha, \beta]_P + [\alpha, \beta]_{\Omega}$$

where $[\cdot, \cdot]_{P+\Omega}$ and $[\cdot, \cdot]_{\Omega}$ are the brackets (72) defined by the bivector fields $P + \Omega$ and Ω, respectively, and $[\cdot, \cdot]_P$, also given by (72), is the Lie bracket on $\Gamma(A^*)$.

If Y is any section of D then, from (89) we deduce

$$\langle [\alpha, \beta]_P + [\alpha, \beta]_{\Omega}, Y\rangle = \big\langle \beta, [Y, \Omega^{\#}\alpha] - \mathcal{L}^W_{*\alpha} Y - \Omega^{\#}\big(\mathcal{L}^{\phi}_Y \alpha\big)\big\rangle \tag{94}$$

where $[\cdot, \cdot]_P$ plays the role of $[\cdot, \cdot]_*$. Taking into account that $\langle [\alpha, \beta]_P, Y\rangle = -\big\langle \beta, \mathcal{L}^W_{*\alpha} Y\big\rangle$ we obtain, for any $\Omega \in \Gamma(\bigwedge^2 A)$,

$$\langle [\alpha, \beta]_{\Omega}, Y\rangle = \big\langle \beta, [Y, \Omega^{\#}\alpha] - \Omega^{\#}\big(\mathcal{L}^{\phi}_Y \alpha\big)\big\rangle. \tag{95}$$

For $P + \Omega \in \Gamma(\bigwedge^2 A)$, (95) becomes

$$\langle [\alpha, \beta]_{P+\Omega}, Y\rangle = \big\langle \beta, [Y, (P + \Omega)^{\#}\alpha] - (P + \Omega)^{\#}\big(\mathcal{L}^{\phi}_Y \alpha\big)\big\rangle$$

or, equivalently,

$$\langle[\alpha,\beta]_{P+\Omega}, Y\rangle = \left\langle\beta, \left(\mathcal{L}^{\phi}_{Y}(P+\Omega)\right)^{\#}\alpha\right\rangle. \tag{96}$$

Then we conclude that

$$[\alpha,\beta]_{P+\Omega} = [\alpha,\beta]_P + [\alpha,\beta]_\Omega \in \Gamma(D^\perp) \quad \Leftrightarrow \quad \mathcal{L}^{\phi}_{Y}(P+\Omega) = 0\,(\mathrm{mod}\, D).$$

□

Acknowledgments

The authors wish to thank the referees for their comments that helped to improve the contents of the paper. Both authors were partially supported by CMUC-FCT.

References

[1] Courant T J 1990 Dirac manifolds *Trans. Am. Math. Soc.* **319** 631–61
[2] Courant T J and Weinstein A 1988 Beyond Poisson structures *Séminaire Sud-Rhodanien de géométrie,* (*Travaux en Cours* vol 27) (Paris: Hermann) pp 39–49
[3] Dorfman I 1993 Dirac structures and integrability of nonlinear evolution equations *Nonlinear Science: Theory and Applications* (Chichester: Wiley)
[4] Grabowski J and Marmo G 2001 Jacobi structures revisited *J. Phys. A: Math. Gen.* **34** 10975–90
[5] Grabowski J and Marmo G 2003 The graded Jacobi algebras and (co)homology *J. Phys. A: Math. Gen.* **36** 161–81
[6] Iglesias D and Marrero J C 2001 Generalized Lie bialgebroids and Jacobi structures *J. Geom. Phys.* **40** 176–200
[7] Iglesias D and Marrero J C 2002 Lie algebroid foliations and $\mathcal{E}^1(M)$-Dirac structures *J. Phys. A: Math. Gen.* **35** 4085–104
[8] Iglesias-Ponte D and Marrero J C 2002 Generalized Lie bialgebroids and strong Jacobi-Nijenhuis structures *Extracta Math.* **17** 259–71
[9] Iglesias D and Marrero J C 2003 Generalized Lie bialgebras and Jacobi structures on Lie groups *Israel Math. J.* **133** 285–320
[10] Kerbrat Y and Souici-Benhammadi Z 1993 Variétés de Jacobi et groupoïdes de contact *C. R. Acad. Sci. Paris,* I **317** 81–6
[11] Kosmann-Schwarzbach Y 1995 Exact Gerstenhaber algebras and Lie bialgebroids *Acta Appl. Math.* **41** 153–65
[12] Kosmann-Schwarzbach Y 1996 The Lie bialgebroid of a Poisson-Nijenhuis manifold *Lett. Math. Phys.* **38** 421–8
[13] Kosmann-Schwarzbach Y and Magri F 1990 Poisson-Nijenhuis structures *Ann. Inst. Henri Poincaré* A **53** 35–81
[14] de León M, Marrero J C and Padrón E 1997 H-Chevalley-Eilenberg cohomology of a Jacobi manifold and Jacobi-Chern class *C. R. Acad. Sci., Paris,* I **325** 405–10
[15] Lichnerowicz A 1978 Les variétés de Jacobi et leurs algèbres de Lie associées *J. Math. Pures Appl.* **57** 453–88
[16] Liu Z 2000 Some remarks on Dirac structures and Poisson reductions*Poisson Geometry* (Warsaw: Banach Center Publications) vol 51 pp 165–73
[17] Liu Z, Weinstein A and Xu P 1997 Manin triples for Lie bialgebroids *J. Diff. Geom.* **45** 547–74
[18] Mackenzie K 1987 *Lie Groupoids and Lie Algebroids in Differential Geometry* (Cambridge: Cambridge University Press)
[19] Mackenzie K and Xu P 1994 Lie bialgebroids and Poisson groupoids *Duke Math. J.* **73** 415–52
[20] Marrero J C, Monterde J and Padrón E 1999 Jacobi-Nijenhuis manifolds and compatible Jacobi structures *C. R. Acad. Sci. Paris,* I **329** 797–802
[21] Nunes da Costa J M 1998 Compatible Jacobi manifolds: geometry and reduction *J. Phys. A: Math. Gen.* **31** 1025–33
[22] Nunes da Costa J M 2002 A characterization of strict Jacobi-Nijenhuis manifolds through the theory of Lie algebroids *Rep. Math. Phys.* **50** 339–47
[23] Roytenberg D 1999 Courant algebroids, derived brackets and even symplectic manifolds *PhD Thesis* (Berkeley: University of California) (*Preprint* math.DG/9910078)
[24] Tan Y and Liu Z-J 1998 Generalized Lie bialgebras *Commun. Algebra* **26** 2293–319
[25] Uchino K 2002 Remarks on the definition of a Courant algebroid *Lett. Math. Phys.* **60** 171–5
[26] Wade A 2000 Conformal Dirac structures *Lett. Math. Phys.* **53** 331–48